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K^--Meson Parity from the Reaction $K^-+d\to\Lambda^0+p+\pi^-$.

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Summary — The possibility of determining the product parity of K^- and Λ^0 relative to the nucleon from the reaction $K^-+d\to\Lambda^0+p+\pi^-$ is discussed. The considerable difference between triplet and singlet Λ^0-nucleon interactions allows to obtain predictions, for the spectrum of the Λ^0-p relative energy, rather strongly depending on the assumed K^--Λ^0 relative parity.

1. – Introduction.

It is not easy to obtain information on the parity of the K^--meson relative to the hyperons from the study of the elementary interaction K^--meson-nucleon. It might be easier to study, for this purpose, in analogy with what happened in the case of π-mesons, the absorption of K^- in light nuclei. DALITZ [1] has already shown that the reaction $K^-+{}^4He\to{}^4H_\Lambda+\pi^0$ is forbidden if the spin value of ${}^4H_\Lambda$ (ground state) is zero and the K^- meson is scalar. We shall here be concerned with the reaction

$$K^-+d\to\Lambda^0+p+\pi^-, \tag{1}$$

which has already been studied experimentally by the Alvarez group [2], in order to investigate whether one can obtain information on the K^- parity

[1] R. H. DALITZ and B. W. DOWNS: *Phys. Rev.*, **111**, 967 (1958).
[2] See R. D. TRIPP: *Proc. Ann. Int. Conf. on High Energy Physics at CERN* (1958), p. 184.

from this reaction. This possibility seems to exist owing to the fact that, under favourable conditions, the parity of the K^- meson will largely determine in reaction (1) the relative weight of the triplet and singlet Λ^0-nucleon states, independently of the details of the elementary K-Λ^0-N interaction. Since the Λ^0-nucleon interaction is expected to be considerably different in singlet and triplet states, one may hope that the desired information could be obtained from the study of those features of reaction (1) which are more dependent on the final state interaction of Λ^0 and nucleon. This feature of the Λ^0-nucleon interaction seems to have been well established by the work of DALITZ and DOWNS [1] on the binding energy of hypernuclei. They have shown that experimental data lead to the determination of a well depth for the triplet potential less than one half than the well depth for the singlet potential. From these data also the low energy scattering parameters (scattering lenght and effective range) can easily be determined. In the present work we investigate in detail the possibility outlined above. Our calculations show that a determination of the K^- parity seems possible from the study of the spectrum of the Λ^0-p relative energy.

2. – Calculations.

The matrix element for the reaction (1) will be calculated in the impulse approximation, that is, we consider that the reaction (1) takes place via the reaction

$$K^- + n \to \Lambda^0 + \pi^- . \tag{2}$$

Such an approximation seems to be justified by the following considerations *a*) the range of the K-nucleon interaction is probably small compared with the dimension of the deuteron [3]; *b*) the multiple π scattering effects are very small since the π-nucleon system must be in the $T = \frac{1}{2}$ isotopic spin state [4]. On the other hand experimental evidence [5] has been accumulated that the reaction (1) can occur also as a two step process:

$$\begin{cases} K^- + (p+n) \to \pi^- + (\Sigma^+ + n) \to \pi^- + \Lambda^0 + p , \\ \text{or} \\ \qquad\qquad \to \pi^- + (\Sigma^0 + p) \to \pi^- + \Lambda^0 + p . \end{cases} \tag{3}$$

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[3] A. PAIS and S. B. TREIMAN: *Phys. Rev.*, **107**, 1396 (1957).

[4] A. FUJII and R. E. MARSHAK: *Nuovo Cimento*, **8**, 643 (1958).

[5] N. HORWITZ, D. MILLER, J. J. MURRAY, M. SCHWARTZ and H. D. TAFT: *Bull. Am. Phys. Soc.*, II, **3**, 363 (1958), Abstract E10.

But if one considers events in which the energy of the emitted π is sufficiently high the probability that the reaction (1) has occurred via the process (3) is certainly very small.

Therefore the matrix element of the reaction will be (we put $\hbar = c = 1$):

$$F = \int \psi^*(\boldsymbol{r}) \exp\left[i \frac{m}{m+M} (\boldsymbol{k} - \boldsymbol{q}) \cdot \boldsymbol{r}\right] T \varphi(\boldsymbol{r}) \, \mathrm{d}\boldsymbol{r} ,$$

wnere T is the transition operator for the reaction (2); $\varphi(\boldsymbol{r})$ is the wave function of the deuteron ground state, $\psi(\boldsymbol{r})$ is the final Λ^0-proton wave function, k and q are the momentum, in the lab. system, of the K^--meson and π-meson respectively, m and M the nucleon and hyperon masses. The dependence on the spin coordinate is understood. The operator T must transform like a scalar if the product parity of the K^- and Λ^0 relative to the nucleon is odd (pseudoscalar K^-), like a pseudoscalar if the product parity is even (scalar K^-). The experimental evidence on the scattering and absorption of K^- in hydrogen at low energy ([6]) (< 100 MeV) strongly indicates that the dominant interaction is in S-states. Therefore in this energy range the operator T, as already noticed by PAIS and TREIMAN ([3]) can be written:

$$T = A(\sigma q) , \qquad \text{for } K^- \text{ scalar},$$

$$T = B , \qquad \text{for } K^- \text{ pseudoscalar},$$

with A and B constant or possibly functions of q^2 an k^2.

Then for K^- scalar the Λ^0-proton system can be produced in singlet and triplet states; for K^- pseudoscalar the Λ^0-proton system is produced only in a triplet state. To be more definite the modulus' square, averaged on the spin variables, of the matrix element F will be

$$|F|^2 = |A|^2 (\tfrac{2}{3} |I_t|^2 + \tfrac{1}{3} |I_s|^2) , \qquad \text{for } K^- \text{ scalar},$$

$$|F|^2 = |B|^2 |I_t|^2 , \qquad \text{for } K^- \text{ pseudoscalar},$$

where

$$I_{t,s} = \int \psi^*_{t,s}(\boldsymbol{r}) \exp\left[i \frac{m}{m+M} (\boldsymbol{k} - \boldsymbol{q}) \cdot \boldsymbol{r}\right] \varphi(\boldsymbol{r}) \, \mathrm{d}\boldsymbol{r} .$$

Due to the considerable difference between triplet and singlet Λ^0-nucleon interaction ([1]) one can expect that the spectrum and angular distribution of emitted particles will be rather sensitive to the K^- parity. However, as observed

([6]) G. ASCOLI, R. D. HILL and T. S. YOON: *Nuovo Cimento*, **9**, 813 (1958); J. D. JACKSON, D. G. RAVENHALL and H. W. WYLD jr.: *Nuovo Cimento*, **9**, 834 (1958).

by FUJI and MARSHAK [4], the π-meson energy spectrum is highly peaked at high energies, therefore a deformation of this spectrum due to the Λ^0-p interaction could not be experimentally detectable. We have turned instead our attention to the spectrum of the Λ^0-p relative energy [7] at a fixed angle of the emitted π. The differential cross section, in the lab system, in this case will be:

$$(4) \qquad \frac{\mathrm{d}\sigma}{\mathrm{d}E_p \mathrm{d}\Omega_q} = \frac{mM}{m+M} \cdot \frac{pE_k}{(2\pi)^5 k} \int |F|^2 q^2 \,\mathrm{d}q \,\mathrm{d}\Omega_p \cdot$$

$$\cdot \delta\left(\varepsilon + \omega_q + \frac{|\boldsymbol{k}-\boldsymbol{q}|^2}{2(m+M)} + \frac{p^2(M+m)}{2Mm} - E_k + M - m\right),$$

where ε is the deuteron binding energy, $E_k = \sqrt{k^2 + m_K^2}$ $\omega_q = \sqrt{q^2 + \mu^2}$. Even if A and B are functions [8] of q^2 the main variation of $|F|^2$ as a function of p^2 is given from the p^2 dependence of I_s and I_t since q^2 varies very slowly as a function of p^2 at fixed π angle. This means that in these conditions we are able to obtain the energy variation of the cross section without requiring the knowledge of the quantities A and B, which depend on the features of the elementary interaction. We will concentrate our attention on those values of E_p which are sufficiently small that only the S wave of the relative motion be distorted.

In this energy range the S phase-shifts can be calculated in the shape independent approximation.

DALITZ and DOWNS [1] give a singlet attractive Yukawa potential with a well depth 0.90 for a range of 0.4 fermi ($= 1/m_K$) and 0.85 for a range of 0.7 fermi ($= 1/2m_\pi$). Since we expect the most important part of potential to arise from the two π-mesons exchange we choose for our calculations the value 0.7 fermi.

The well depth for triplet potential is also attractive but smaller than the singlet one by a factor $\frac{1}{3}$ (for a range of 0.7 fermi). For such a low value of the well depth the triplet scattering length a_t result much smaller than the singlet one a_s ($a_t = 1/16\ a_s$); therefore we can neglect completely the final state triplet interaction.

For the singlet state we assume a wave function of the from

$$\psi_s(\boldsymbol{r}) = \exp[i\boldsymbol{p}\cdot\boldsymbol{r}] + \frac{\exp[-i\delta_s]}{pr} \sin\delta_s (\exp[-ipr] - \exp[-\eta r]) ,$$

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[7] By spectrum of relative energy we mean the number of events in the interval $\mathrm{d}E_p$ at the value $E_p = [(m+M)/2Mm]p^2$ of the relative kinetic energy.

[8] According to PAIS and TREIMAN (see ref. [3]) A and B are almost independent of q^2.

where the term $\exp[-\eta r]$ is added in order to approximate the exact wave function inside the range of the Λ-N forces. η is determined by the condition that the S part of the wave function gives the correct effective range at $p = 0$; hence η is defined by the relation

$$r_0 = \frac{3}{\eta} - \frac{4}{a\eta},$$

where a is the scattering lenght and r_0 the effective range as defined by BLATT and JACKSON (9).

Using as wave function of the deuteron ground state the Hulten function

$$\varphi(r) = C \frac{\exp[-\alpha r] - \exp[-\beta r]}{r},$$

with

$$C = \left(\frac{\alpha+\beta}{\alpha-\beta}\right)\left(\frac{\alpha\beta}{2\pi(\alpha+\beta)}\right)^{\frac{1}{2}}, \qquad \alpha = 2.31 \cdot 10^{12}\ \mathrm{cm}^{-1}, \quad \beta/\alpha = 6.35,$$

the cross-section (4) will be given, omitting for semplicity constant factors, by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E_p\,\mathrm{d}\Omega_q} \propto p\,|A|^2\left(\frac{2}{3}|I_t|^2 + \frac{1}{3}|I_s|^2\right)\frac{E_K}{k}, \qquad \text{for scalar } K^-,$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E_p\,\mathrm{d}\Omega_q} \propto p\,|B|^2\,|I_t|^2\,\frac{E_K}{k}, \qquad \text{for pseudoscalar } K^-,$$

where

$$|I_s|^2 \propto \left\{ \frac{1}{[\alpha^2 + (l-p)^2][\alpha^2 + (l+p)^2]} + \frac{1}{[\beta^2 + (l-p)^2][\beta^2 + (l+p)^2]} - \right.$$

$$- \frac{1}{2lp(\beta^2 - \alpha^2)} \ln z - \frac{\ln^2 z}{16 l^2 p^2} + \frac{1}{4l^2p^2}\left[\frac{\cos\delta_s}{2}\ln z + \right.$$

$$+ \sin\delta_s\left(\operatorname{arctg}\frac{l+p}{\alpha} + \operatorname{arctg}\frac{l-p}{\alpha} - \operatorname{arctg}\frac{l+p}{\beta} - \operatorname{arctg}\frac{l-p}{\beta} - \right.$$

$$\left.\left.\left. - 2\operatorname{arctg}\frac{(\beta-\alpha)l}{(\alpha+\eta)(\beta+\eta)+l^2}\right)\right]^2\right\} \frac{q^2}{(q/\omega_q) + (q - k\cos\theta_q)/(m+M)},$$

and

$$z = \frac{\alpha^2 + (l+p)^2}{\alpha^2 + (l-p)^2}\,\frac{\beta^2 + (l-p)^2}{\beta^2 + (l+p)^2}, \qquad l = \frac{m}{m+M}|\boldsymbol{k} - \boldsymbol{q}|;$$

$|I_t|^2$ follows from the preceding expression by putting $\delta = 0$.

(9) J. M. BLATT and J. D. JACKSON: *Phys. Rev.*, **76**, 18 (1949).

3. – Numerical results and conclusions.

We have calculated the scattering legth and the effective range for the Yukawa well with the interpolation formulas given by BLATT and JACKSON (9). With a well depth $S=0.85$ and intrinsic range $b=2.1196\cdot 0.7$ fermi one obtains

$$a_s = -5.60\cdot 10^{-13}\ \text{cm}\,, \qquad r_{0s} = 1.83\cdot 10^{-13}\ \text{cm}\,, \qquad \eta = 1.85\cdot 10^{13}\ \text{cm}^{-1}\,.$$

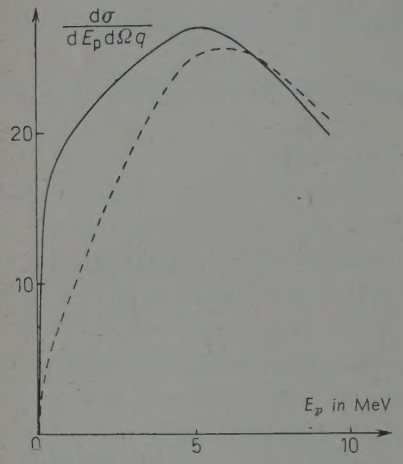

Fig. 1.

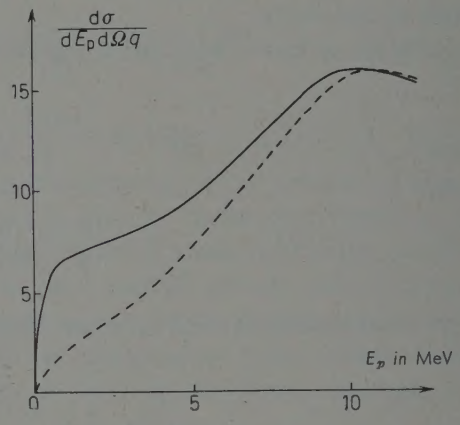

Fig. 2.

Our calculations have been performed for an incident K-meson energy of 30 MeV. The result of our calculation for the angles 30° 60° 90° 150° of the emitted π are shown in Fig. 1, 2, 3, 4 in the same arbitrary units. Since the

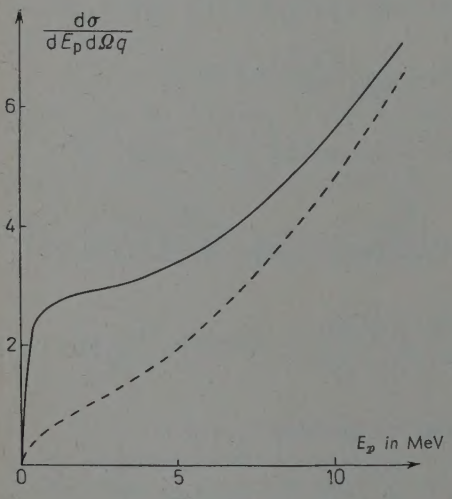

Fig. 3.

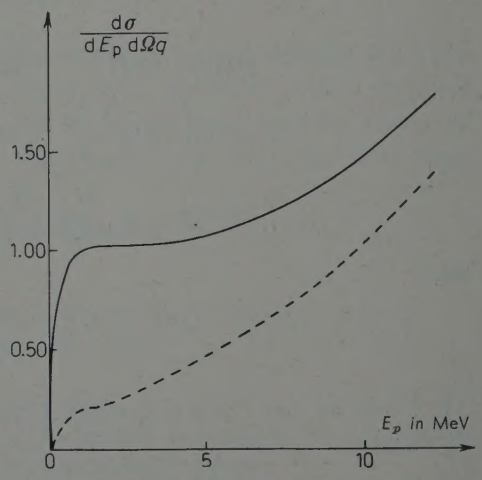

Fig. 4.

absolute value of the cross section is not known in order to discriminate between the two possibilities (K^- scalar or pseudoscalar) one should compare the relative number of events in different energy ranges. For instance at 90° (which seems to us the most favourable from the experimental point of view) the ratio between the number of events in the range $0 < E_p <$ < 5 MeV and the number in the range $5 < E_p < 12$ MeV is 1/2.36 for the scalar case 1/5.25 for the pseudoscalar case. At greater angles the difference between the two cases is even larger but the intensity decreases considerably. We have also plotted the cross-section integrated on the variable E_p up to $E_p = 5$ MeV as a function of the angle of the emitted π (Fig. 5). Also in this plot the scalar and pseudoscalar cases differ considerably. We point out that for the determination of p (in order to obtain E_p in terms of which the cross-sections are plotted) the knowledge of k, q and the momentum of either the proton or the Λ^0 is necessary. These measurements seem feasible, even if the interesting values of p are rather small, because the problem is overdetermined and internal consistency checks are possible.

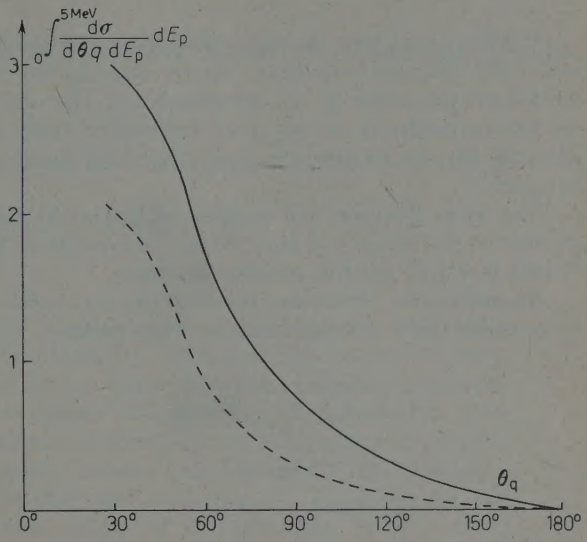

Fig. 5.

It seems therefore that the proposed method should be adequate for the determination of the K^--meson parity.

* * *

We are grateful to Prof. R. GATTO for calling our attention to this problem; useful discussions with him and Prof. M. CINI are acknowledged.

Note added in proof.

More recently DALITZ (*) has refined his analysis of hypernuclear binding energies and found the value $S = 0.75$ for the well-depth parameter of the Λ^0 nucleon potential

(*) R. H. DALITZ: *Proc. of the Conference on High Energy Physics at CERN* (Geneva, 1958).

in the singlet state, whether the range is 0.4 or 0.7 fermi. Therefore, according to the above value of S and with the range of 0.7 fermi as already assumed, the following quantities are so modified:

$$a_s = -3.12 \cdot 10^{-13} \text{ cm}, \qquad r_{0s} = 2.12 \cdot 10^{-13} \text{ cm}, \qquad \eta = 1.76 \cdot 10^{13} \text{ cm}^{-1},$$

Obviously in the preceding diagrams the behaviour of the cross-section for the scalar K^- changes somewhat. As an example we have plotted in Fig. 3*a* the cross-section for the scalar K^- as calculated with the new data (full line), at $\theta_q = 90°$, which we have considered as the most favourable from the experimental viewpoint. The cross-section for the pseudo scalar K^-, unchanged, (dashed line) is also plotted for comparison.

The ratio between the number of events in the range $0 < E_p < 5$ MeV and the number in the range $5 < E_p < 12$ MeV turns out to be 1/2.93 for the scalar case, while it still is 1/5.25 for the pseudoscalar case.

Therefore the distinction between the two cases remains relevant and worth-while to consider from the experimental view-point.

RIASSUNTO

Si discute la possibilità di determinare la parità prodotto del K^- e del Λ^0 relativa al nucleone dalla reazione $K^- + d \to p + \Lambda^0 + \pi^-$. La considerevole differenza tra le interazioni Λ^0-nucleone nello stato di tripletto e singoletto permette di ottenere previsioni sullo spettro dell'energia relativa Λ-p le quali dipendono sensibilmente dal segno che si assume per la parità relativa K^--Λ^0.

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Ferroelectric Properties of a Material Made of Titanium Oxide.

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(ricevuto il 21 Febbraio 1959)

Summary. — A ceramic-type material was prepared from TiO_2 by a special technique described elsewhere (1). Although this material is very similar to rutile, ist properties are quite different. Suitable specimens were prepared with a view to prove the existence of ferroelectric properties, and readings were taken, as shown in the accompanying graphs: 1) The variations of the capacity against bias voltage; 2) the hysteresis loops with varying temperatures and voltages; 3) variations in the dielectric constant in a wide range of temperatures, from -60 °C to $+350$ °C. These results prove the existence of a ferroelectric phenomenon. It is supposed that it should be possible to use this material as a test material for further research on *matter in the solid state.*

1. – Introduction.

Some special materials, the so-called ferroelectric materials, have aroused considerable interest owing to the possibilities of their being largely applied in electronics and electroacustics, and because they pose very foundamental problems in solid state physics.

The chief characteristic of these materials is that all their properties change at the Curie point, as might be expected from thermodynamical calculations. These changes at the ferroelectric transition point, due to space lattice distortion and similar to, but much greater than, the corresponding changes caused by magnetic influence, were first observed in a group of substances having the crystalline structure of perovskite.

A report appeared in *Nature* in 1952 (1), where mention was made that

(1) L. Nicolini: *Nature*, **170**, 938 (1952).

a ceramic-type material had been obtained by high temperature (about 1400 °C) air sintering of pieces of titanium oxide which had been processed in the same way as the materials used in the preparation of ceramic substances. This report was later (Nov., 1952) amplified and presented before the Académie de France ([2]). The preparation of this material was the object of Patent No. 7592 ([3]) and published in October 1954 ([4]).

X-ray analysis by the Debye and Sherrer method—chamber circumference 240 mm and radiation $K_{\alpha}Cu$—showed the same rectangular axis ratio (1:0.911) as pure rutile.

In the above-cited report, values of ε were quoted for different frequencies and temperatures. These showed that the dielectric constant of the material was very high for frequencies at room temperature, and much higher at other temperatures. With frequencies of 10^5 to 10^7 (where dipolar polarization does not take place), ε has values equal to those of normal rutile. There was therefore the temptation to explain this phenomenon by supposing that polarization takes place in a domain structure; and since there was a correspondence with the behaviour of ferromagnetic materials the existence of ferroelectric properties was suspected ([5]).

This research was taken up again and samples were prepared in order to verify the ferroelectric properties of this material and to ascertain in consequence:

a) the variation of dielectric polarization according to the direct current,

b) the existence of a hysteresis loop similar to the magnetic hysteresis loop,

c) the existence of Curie points at which the crystal structures change.

Small discs were cut from the sintered material with a diamond saw. They were silvered on both faces, after these had been abrasively polished to a perfectly smooth and uniform surface. Having been thus prepared, the samples were placed between two spring contacts in a special stand.

2. – Variation of dielectric polarisation as function of the superimposed current.

Since ferroelectric materials show a considerable variation in the dielectric constant with every variation in the electric field they are subjected to, so that this property is designated by the term « ferroelectricity » by analogy

([2]) L. NICOLINI: *Lettre sous plis cachetés* (November 29th, 1952).
([3]) Italian Patent 7592, (Aug. 9th, 1951).
([4]) L. NICOLINI: *Rendiconti A.E.I.*, Bellagio, 31 October 1954, n. 176.
([5]) L. NICOLINI: *Minutes of the Faculty of Engineering*, **70**, no. 50 (1952).

with ferromagnetic phenomena, experiments were made with some specially prepared specimens in order to observe the « non-linearity ».

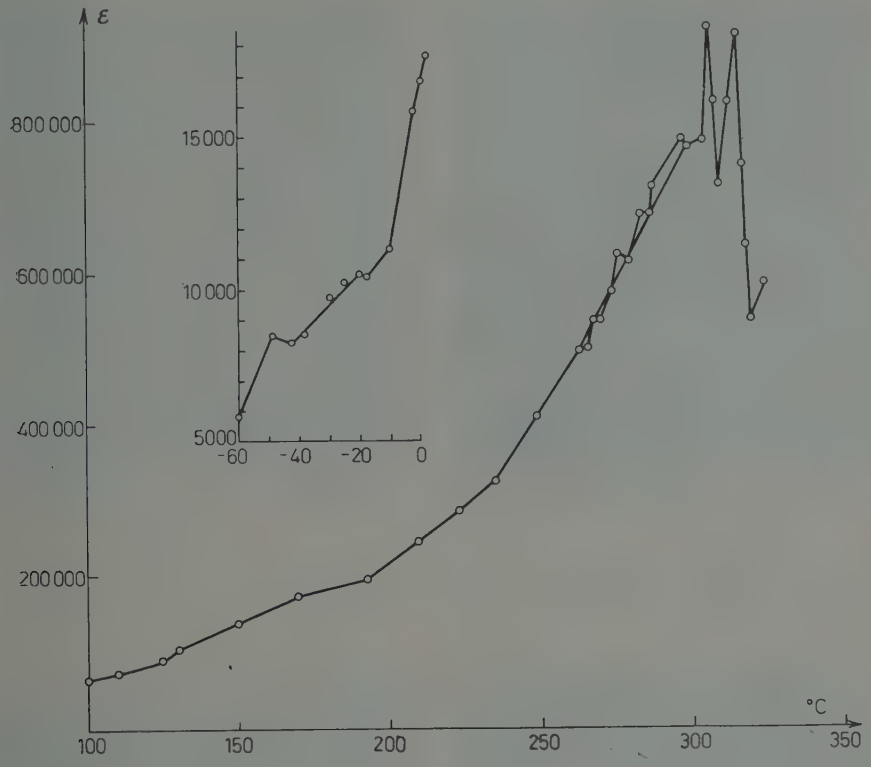


Fig. 1.

The differential capacity $C = \partial Q / \partial V$ was measured by a technique similar to that adopted by A. WADNJAL (6). Measurements were made by applying to the condenser (with a dielectric of prepared material) a small sinusoidal voltage, frequency 100 Hz, superimposed to a d.c. bias voltage. Values of C are related to the value C_0, obtained for $V_c = 0$; this bears out the hysteresis loops and shows up the « non-linearity ». Readings were also taken in vacuum and in air.

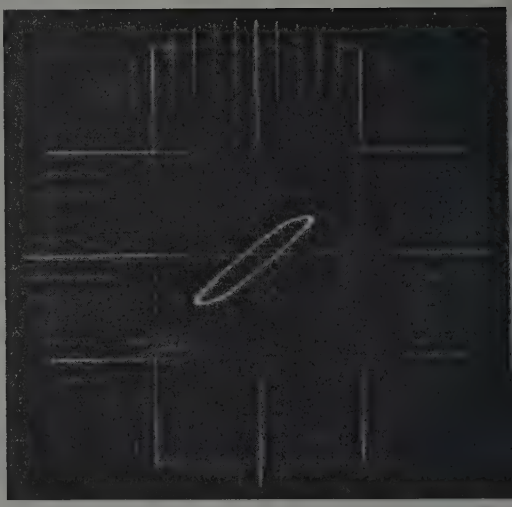
The differences between readings in air and readings in vacuum are due to the porosity of the material and its high sensitivity to small differences in temperature and humidity.

(6) M. VADNJAL: *A.E.I. Congress* (Bellagio, October 1954).

$V = 30$ V

1

0 °C



2

60 °C

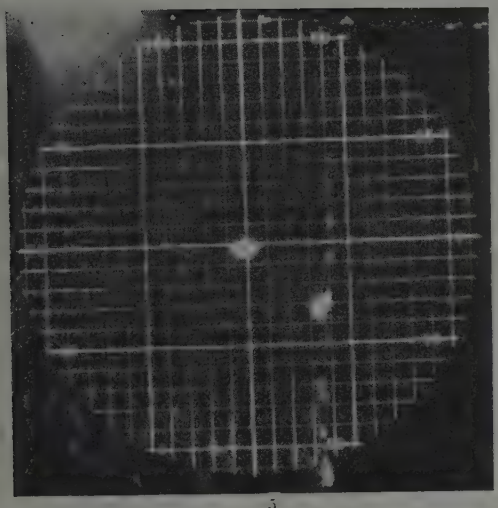
3

120 °C

4

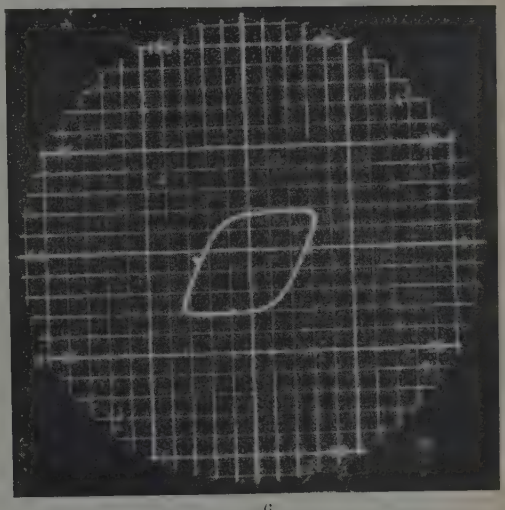
180 °C

$T = 18\,^{\circ}\mathrm{C}$



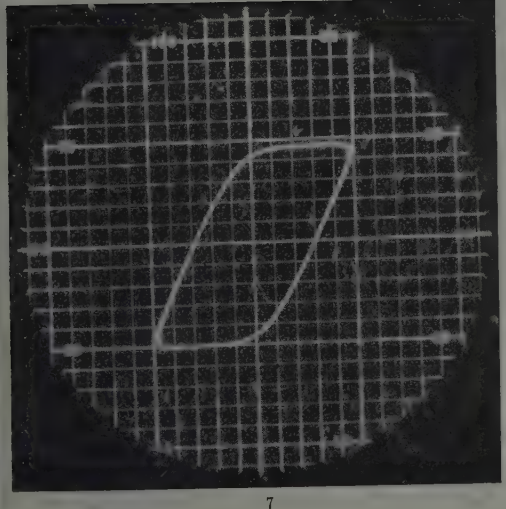
5

75 volt



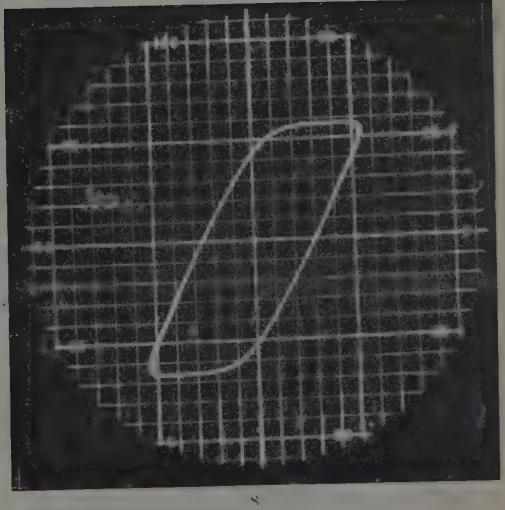
6

65 volt



7

100 volt



8

120 volt

3. – Dielectric hysteresis.

As the hysteresis loops of ferroelectric materials prove that the direction of polarization and domain alignment can be altered by an applied electric field; readings were taken on a silvered specimen 1 cm² in area by 1 mm thick, at 30 V between 0° and 100 °C. A cathode ray oscillograph was used for this, with the simple circuit first described by SAWYER and TOWER (7). Hysteresis loops were also plotted with a temperature constant at 18 °C, but varying the voltage from 0 to 120 V (see Fig. 2).

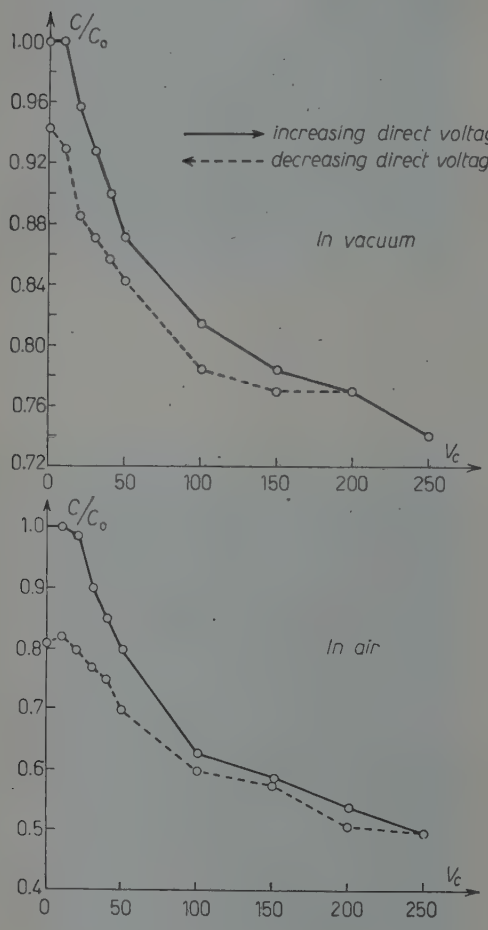


Fig. 3.

Research here was limited by the high loss due to conductivity caused by impurities and structural faults (8).

4. – Variation of the dielectric polarisation according to the temperature: Curie point.

The relation of the dielectric constant to the temperature is shown in Fig. 3. Readings were taken at temperatures ranging from — 60 °C to + 350 °C. It is clear from this curve that capacity increases considerably with temperature, and that the upper Curie point is about 350 °C.

There are two temperatures, — 10 °C and — 60 °C, which might be two more Curie points, but a crystallographic confirmation of this is needed.

The spontaneous polarization is accompanied by a spontaneous strain and piezoelectric and elastic anomalies accompany the die-

(7) C. B. SAWYER and C. N. TOWER: *Phys. Rev.*, **35**, 269 (1930).

(8) V. ANDRESCIANI, L. NICOLINI and D. SETTE: *Note Recensioni Notizie*, 4 Luglio 1957.

lectric anomaly; these marked effects which take place at such high temperatures ([9]) in a material of this type cause the readings to vary slightly even in the short time needed for a single reading, and thus they are by no means definitive. They were, however, repeated several times, and they continued to show two Curie points; though these were also shown by Rochelle salt, but at very low temperature (— 20 °C and + 20 °C) ([10]).

A crystallographic study will be made of these points, and this will be reported elsewhere.

On the basis of all these measurements, I am entitled to think that there is much evidence to bear out the theory of there being a *ferroelectric phenomenon* in the prepared material, and therefore that it possesses a special crystal system similar to that of ferromagnetic materials.

Since reversible polarization, which is a feature of ferroelectricity caused by pseudosymmetrical transition, requires at the same time a change in the homopolar bond system of the different kinds of atoms making up the structure as has been reported in their paper ([11,12]) by MEGAW *et al.* for WO_3, it would be interesting to make a deeper study of the matter.

The ferroelectric phenomenon already observed in barium titanates and other compounds of the perovskite group, not to mention the other crystals with a very high dielectric constant, does not appear to me to have been observed in TiO_2, which, as VERSWEY and A. D. BUGEL have shown ([13]), has been considered a dielectric with a dielectric constant $\varepsilon = 100$, and by SZIGETI ([14]) a ionic crystal Rutile $\parallel$ $\varepsilon = 89$, Rutile $\perp$ $\varepsilon = 173$.

And the possibility of carrying out new research on *matter in the solid state*, using this particular titanium dioxide, makes the work very interesting.

([9]) The dielestrics with a high Curie point aim Sodium Niobate $(NaCd)NbO_3$, with a Curie temperature of from 100 °C to 300 °C and a dielectric constant of about 10 000; and Lead Metaniobate, $Pb(NbO_3)$, with a Curie point of 570 °C and a dielectric constant of 7 000.

([10]) A. MELMED, F. JONA and R. PEPINSKY: *Ferroelectric and High Dielectric Crystals: Dielectric Properties of* $Na(KNH_4)\,C_4H_4O_6 \cdot 4H_2O$ *Mixed Crystals*, Technical Report no. 20, Contract A.F.33 (616–2133), Wright Air Development Center (January 1954).

([11]) G. SHIRANE, F. JONA and R. PEPINSKY: *Proc. I.R.E.*, **43**, 1738 (1955); H. D. MEGAW: *Acta Crystall.*, **5**, 739 (1952); **7**, 187 (1954); G. GOODMAN: *Journ. Amer. Ceramic Soc.*, **36**, 368 (1953).

([12]) W. L. KEHL, R. G. HAY and D. WAHL: *Phys. Rev.*, **82**, 774 (1951).

([13]) VERSWEY and A. D. BUGEL: *Revue Téchnique* (February 1949).

([14]) F. MATOSSI: *Proc. Ind. Acad. Sci.*, **23**, 279 (1950); *Journ. Chem. Phys.*, **2**, 1543 (1951); B. SZIGETI: *Trans. Faraday Soc.*, **45**, 155 (1949).

RIASSUNTO

Con speciale tecnica descritta altrove [1] è stato preparato un materiale quasi ceramico costituito di TiO_2. Questo materiale che apparentemente è rutilo ha proprietà tutto affatto differenti. Campioni adatti sono stati preparati allo scopo di provare l'esistenza di proprietà ferroelettriche. Sono stati perciò eseguiti, come dimostrano i grafici riportati, misure riguardanti: 1) le variazioni di capacità al variare della tensione impressa; 2) i cicli di isteresi al variare della temperatura e della tensione; 3) la costante dielettrica al variare della temperatura in un vasto campo di temperature cioè da -60 °C a $+350$ °C. Tali misure danno la prova dell'esistenza di un fenomeno ferroelettrico.

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Cloud Chamber Study of Penetrating Showers Underground (*).

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(ricevuto il 9 Marzo 1959)

Summary. — A multiplate cloud chamber containing fifteen lead plates of 1 cm thick was used to observe penetrating showers underground. Fifteen and twenty-three penetrating showers, having four secondary shower particles on the average, have been obtained during 667.9 h and 3 603.1 h at 50 m w. e. and 250 m w. e., respectively. Special attention was paid to distinguish penetrating showers produced by μ-mesons from those by the nucleonic component, the chamber of large width (100 cm) having been set as close to the upper wall in the tunnel as possible. Almost all of the observed showers produced by isolated incident particles are considered as probably produced by μ-mesons (named P-showers phenomenologically), and those by one of two or more incident particles as due to the nucleonic component (named S-showers), since the m.f.p. of the nucleonic component for nuclear interaction is about 10^{-4} times shorter than that of μ-mesons. After correcting for the triggering efficiency of the apparatus, the ratios of frequencies of S-showers to that of P-showers have turned out to be 1.1 ± 0.3 and 0.92 ± 0.23 at both depths, which means that a half of the high energy nuclear interactions underground is produced by the nucleonic component. The depth dependence of frequencies of P-showers is compared with the prediction by Weizsäcker and Williams' treatment of μ-meson interactions. In addition, it has remarkably been observed that P-showers have a characteristic different from that of S-showers, *i.e.*, the average number of heavily ionizing secondaries of P-showers is 0.3 per shower, while the value of S-showers is 2.8 per shower.

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1. – Introduction.

Static characters of μ-mesons have experimentally been studied by the use of accelerators ([1]). Especially the spin of μ-mesons has been determined recently with reference to the parity non-conservation ([2,3]), and the discovery of π-e decays ([4]) has discarded the destruction of the global symmetry of weak interaction, establishing similarity between μ-meson and electron. As a consequence, it may probably be considered that the μ-meson is the same as the electron except their mass difference, but the difference itself providing a fundamental question about μ-mesons.

Dynamic characters of μ-mesons have, on the other hand, not been able to be studied with accelerators in the energy range above several hundred MeV, but only by cosmic rays underground. The so-called anomalous large angle scattering of μ-mesons has recently been proved not to be anomalous up to about 1 GeV/c of μ-meson momentum ([5]), this fact also showing the μ-meson to the be « heavy » electron.

Therefore it is very important to determine whether or not the high energy nuclear interactions of μ-mesons are interpreted only by electromagnetic interactions of μ-mesons, *i.e.*, are reproduced as that of the heavy electron. On the nuclear stars underground several investigations have been made ([6]) since the emulsion work by GEORGE and EVANS ([7]), and have concluded that the nuclear interactions can be interpreted only by the electromagnetic interactions of μ-mesons on the basis of Weizsäcker and Williams' treatment of interaction ([8,9]) with constant γ-π cross-sections. However, on the hard showers

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([1]) See, for example, J. RAINWATER: *Ann. Rev. Nucl. Sci.*, **7**, 1 (1957).

([2]) R. L. GARWIN, L. M. LEDERMAN and M. WEINRICH: *Phys. Rev.*, **105**, 1415 (1957).

([3]) T. COFFIN, R. L. GARWIN, L. M. LEDERMAN, S. PENMAN and A. M. SACHS: *Phys. Rev.*, **106**, 1108 (1957).

([4]) T. FAZZINI, G. FIDECARO, A. W. MERRISON, H. PAUL and A. V. TOLLESTRUP: *Phys. Rev. Lett.*, **1**, 247 (1958).

([5]) S. FUKUI, T. KITAMURA and Y. WATASE: *Progr. Theor. Phys.*, **19**, 348 (1958), *Phys. Rev.* **113**, 315 (1959).

([6]) For example, G. N. FOWLER and A. W. WOLFENDALE: *Progress in Elementary Particle and Cosmic Ray Physics*, vol. **4** (edited by J. G. WILSON and S. A. WOUTHUYSEN, Amsterdam, 1957), p. 105.

([7]) E. P. GEORGE and J. EVANS: *Proc. Phys. Soc.* (*London*), A **63**, 1248 (1958); A **68**, 829 (1955).

([8]) C. F. VON WEIZSÄCKER: *Zeits. f. Phys.*, **88**, 612 (1934).

([9]) E. J. WILLIAMS: *Proc. Roy. Soc.*, A **139**, 163 (1933); *Kgl. Dansk. Vid. Selsk.*, **13**, no. 4 (1935).

underground, it has been pointed out by us ([10]) and FOWLER *et al.* ([11]) that the depth dependence of frequencies of hard showers, obtained with counter hodoscopes at 50 m w.e., 200 m w.e. ([12-14]) and 1600 m w.e. ([15]), could not be interpreted with the W-W treatment of electromagnetic interaction of μ-mesons with constant γ-π cross-sections, while there have been reported a few data ([12,16]) which are consistently interpreted only by the electromagnetic interactions of μ-mesons.

However, all of the data, so far presented on the nuclear interaction of μ-mesons, have been obtained in underground experiments, on the assumption that the underground nuclear interactions of high energy are exclusively produced by μ-mesons. The assumption contradicts with our experimental result ([17]), though statistically not enough, showing that part of penetrating showers underground are produced by the nucleonic component. This is the case also for the nuclear stars, since stars are nuclear interactions of lower energy than that of penetrating showers.

Therefore the present experiment has been performed with a cloud chamber with special intention to distinguish the penetrating showers produced by μ-mesons from the underground penetrating showers.

2. – Apparatus and experimental procedure.

A large multiplate cloud chamber was used in combination with five trays of counters. A schematic diagram of the apparatus is shown in Fig. 1. The observing station was located in the Isohama Tunnel of the Government Railway, Yaizu City, Shizuoka, Japan. The observation was performed at 50 m w.e. and 250 m w.e. underground during receptive times of 667.9 hours and 3603.1 hours respectively, for about eighteen months from March, 1956, until November, 1957.

The cloud chamber has an illuminated region of $(80 \times 70 \times 40)$ cm^3 and con-

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([10]) S. HIGASHI, M. ODA, T. OSHIO, H. SHIBATA, K. WATANABE and Y. WATASE: *Progr. Theor. Phys.*, **16**, 250 (1956).

([11]) G. N. FOWLER and A. W. WOLFENDALE: *Nucl. Phys.*, **3**, 299 (1957).

([12]) P. E. ARGAN, A. GIGLI and S. SCIUTI: *Nuovo Cimento*, **11**, 530 (1954).

([13]) S. HIGASHI, M. ODA, T. OSHIO, H. SHIBATA, K. WATANABE and Y. WATASE: *Journ. Phys. Soc. Japan*, **11**, 1021 (1956).

([14]) D. KESSLER and R. MAZE: *Physica*, **22**, 69 (1956).

([15]) P. H. BARRET, M. L. BOLLINGER, G. COCCONI, Y. EISENBERG and K. GREISEN: *Rev. Mod. Phys.*, **24**, 133 (1952).

([16]) D. KESSLER and R. MAZE: *Nuovo Cimento*, **5**, 1540 (1957).

([17]) S. HIGASHI, T. OSHIO, H. SHIBATA, K. WATANABE and Y. WATASE: *Nuovo Cimento*, **5**, 592 (1957).

tains fifteen 1 cm thick lead plates corresponding to about one geometrical mean free path of the nucleonic component in lead. The probability is 1/13 at 50 m w.e. that one could find penetrating particles which pass through the cloud chamber in its sensitive time, regardless of their incident direction, without any relation to triggering events. The same probability is 1/30 at 250 m w.e. These are very small probabilities to misinterpret the pictures of the cloud chamber, considering the distribution in the incident angles of penetrating particles. Photographs of the chamber were taken stereoscopically. Other details about the chamber have been described in reference [18].

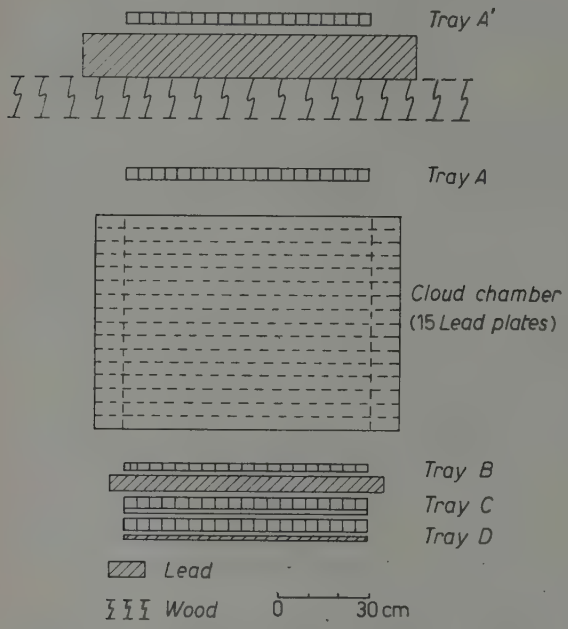

Fig. 1. – Schematic diagram of the apparatus.

Over the chamber a 15 cm thick lead layer was placed, serving as producer of penetrating showers.

Five trays of counters, A', A, B, C and D, selected the events to be recorded. Arrangement of these trays is shown in Fig. 1. Every counter was hodoscoped, and dimensions of the counters are 2 cm in diameter and 45 cm in length for those of tray B, and 4 cm in diameter and 45 cm in length for those of the other trays. The resolving time of the hodoscopes is about 150 μs. Besides seven trays of couters were set a few meters apart from the apparatus described above, serving as shower trays. Since these trays were added mainly in regard to multiple penetrating particles (M.P.P.s) underground [18], further description of these trays is not given here. Only, attention is paid to the fact that all penetrating showers were not accompanied by discharge of any counter of these shower trays.

During a half of the observing time, at 250 m w.e., the chamber was triggered by coincidences « $(A' \geqslant 1)(A \geqslant 1)(B \geqslant 2)(C \geqslant 2)(D \geqslant 1)$ », and during the remaining time by coincidences « $(A' \geqslant 1)(A \geqslant 1)(B \geqslant 2)(C \geqslant 1)(D \geqslant 2)$ ». The

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[18] S. HIGASHI, T. OSHIO, H. SHIBATA, K. WATANABE and Y. WATASE: *Nuovo Cimento*, **5**, 597 (1957).

resolving time of the coincidence was about 15 μs. The same coincidences were, at 50 m w. e. also, preferable to be taken as triggering pulses, since the triggering is required to select the penetrating showers in biasing the events as little as possible. This kind of coincidences occurred, however, too frequently at 50 m w. e. for suitable operation of the cloud chamber. Hence, at 50 m w. e., coincidence pulses « $(A' \geqslant 1)(A \geqslant 1)(B \geqslant 2)(C \geqslant)2(D \geqslant 2)$ » triggered the chamber and associated apparatus. Correspondingly, at 250 m w. e., we selected the events due to coincidences « $(A' \geqslant 1)(A \geqslant 1)(B \geqslant 2)(C \geqslant 2)(D \geqslant 2)$ », from the data taken as described above, with the help of hodoscope patterns. This selection caused no trouble for the comparison between the data at both depths.

Single particles « $(A' \geqslant 1)(A \geqslant 1)(B \geqslant 1)(C \geqslant 1)(D \geqslant 1)$ » were registered at both depths. The rates were (12.0 ± 0.25) counts/min at 50 m w. e. and (0.54 ± 0.07) counts/min at 250 m w. e., and these are consistent with the expected values from the depth-intensity curve.

The whole apparatus of the present experiment was settled on a level as high as possible in the tunnel, so that secondary penetrating particles produced in the ground above the tunnel can fall onto the cloud chamber as bundle of particles before diverging into isolated particles, the distance between the upper wall of the tunnel and the chamber being about 150 cm. The large size of the chamber, 100 cm width and 40 cm depth, serves also for the same purpose (*). If this consideration for setting were not taken into account, then a secondary nucleonic component would frequently be mistaken as a primary μ-meson, since any association of the other particles cannot be found in the same chamber after the secondary particles were isolated from one another.

3. – Experimental results.

3·1. *Classification of penetrating showers.* – Photographs of the chamber were taken by the selection of coincidences « $(A' \geqslant 1)(A \geqslant 1)(B \geqslant 2)(C \geqslant 2)(D \geqslant 2)$ », and are classified by the number of the incident penetrating particles, which are tabulated in Tables I and II (**).

(*) The chamber used by KESSLER *et al.* [16] has dimension of about 56 cm square and 85 cm height, and this consideration for setting seems not to have been taken into account.

(**) In the tables, photographs having no incident penetrating particles are those in which there are a little more electrons than are in photographs taken by random triggering. Or otherwise they are those each showing a dense electron cascade having entered the chamber through its side faces. Almost all the events containing one incident

TABLE I. – *Classification of photographs according to the number of incident penetrating particles at* 50 m w. e.

Number of incident penetrating particles	Number of photographs	Penetrating showers	
		in C.C.	above C.C.
0	56	0	0
1	511	7	0
2	230	2	2
3	48	4	3
4	19	3	3
5	4	0	1
6	6	1	3
7	5	1	2
—	—	—	—
10	1	1	1

TABLE II. – *Classification of photographs according to the number of incident penetrating particles at* 250 m w. e.

Number of incident penetrating particles	Number of photographs	Penetrating showers	
		in C.C.	above CC.
0	168	0	0
1	757	12	0
2	52	1	2
3	16	3	7
4	13	3	7
5	2	0	2
6	5	3	2
7	1	0	0
8	2	1	0
—	—	—	—
12	1	0	0
—	—	—	—

penetrating particle are those in which a single penetrating particle enters the chamber and produces a small electron cascade in a lead absorber below the chamber. In this case, also, there are contained penetrating showers produced by a single penetrating particle in a lead plate in the chamber, which are phenomenologically named *P*-showers. Photographs with two or more incidents, on the other hand, are those representing secondary penetrating particles which were produced above the chamber and penetrated through it. In these events there are found a number of nuclear interactions, both stars and penetrating showers, the latter being named *S*-showers.

In addition, M.P.P.'s [19] are contained in the cases of two or more incident penetrating particles, but description about them will be given in another paper.

The « penetrating showers » are defined to be such events from which two or more penetrating particles originated. The « penetrating particle » means the particle which penetrates two or more lead plates (1 cm thick) in the chamber with neither multiplication nor remarkable scattering, and produces minimum ionizing tracks in the chamber. The penetrating showers are classified into two groups, P-showers and S-showers. P-showers are penetrating showers whose parent is associated with no other penetrating particle, and S-showers are those whose parent is accompanied by other penetrating particles. Their schematic features are shown in Fig. 2. As a first approximation, it is quite possible to consider that P-showers are the showers produced by μ-mesons and S-showers are the ones produced by the nucleonic component which are ejected from the showers that occurred above the chamber, since the secondary nucleonic component can interact with probabilities about 10^4 times larger than that of μ-mesons and since the nuclear interactions of secondary π-mesons occurred much more frequently than the π-μ decays in the ground. In order to analyse the present data in connection with some data taken hitherto with counter hodoscopes, and to compare directly the features of P-showers with those of S-showers, attention must be paid to the fact that S-showers have larger probabilities to be recorded than P-showers. This is because, in the case of S-showers, accompanying penetrating particles can also give rise to the triggering of the apparatus. Hence S-showers were reclassified into the following two subgroups; S_t-showers are those in which secondary penetrating particles of the shower formed in the chamber as well as the parent particles have contributed to the triggering of the apparatus, and S_n-showers those for which triggering required the entrance of accompanying penetrating particles in addition to the primary and the secondary particles. Hereafter in comparing the S-showers to the P-showers the S_n-showers will be excluded from our consideration. The S_t-showers, which are induced by the nucleonic component, should be considered to have the same features as those

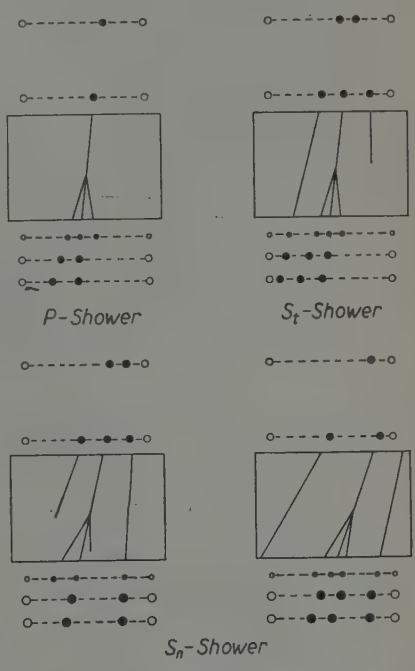

Fig. 2. – Schematic illustration of the P-shower and the S-shower.

of the penetrating showers observed at high altitudes ([19,20]), and hence it is estimated that the present apparatus is triggered by the S_t-showers, where the threshold energy is about 4 GeV.

According to this classification, the data obtained are listed in Table III. It should be remarked that S_t-showers were observed as frequently as P-showers. This fact shows that part of the penetrating showers observed under-

TABLE III. – *Receptive times and observed numbers of penetrating showers at both depths.*

Depth	Receptive time	P-shower	S_t-shower
50 m w. e.	667.9 h	7 showers	8 showers
250 m w. e.	3 603.1 h	12 showers	10 showers

ground, for example by means of nuclear emulsion, must have been produced by secondary nucleonic components.

3·2. *Comparison of P-showers and S_t-showers.* – Comparison of P-showers and S_t-showers, which are observed under the same conditions, may be expected to show the difference between the penetrating showers produced by μ-mesons and the ones produced by nucleonic components.

In Tables IV and V are listed the classifications of P-showers and S_t-showers according to the number of secondary penetrating particles (n_s) and to the number of heavily ionizing secondary particles (N_h). The heavily ionizing particle was distinguished from the penetrating particle at sight without any instrument. This did not, however, cause such ambiguity as to influence

TABLE IV. – *Classification of P-showers according to n_s and N_h.*

N_h \ n_s	2	3	4	5	6	7	8	12	Total
0	2	5	3 (*)	1	1	—	1	—	13
1	1	1	1	1	—	—	1	—	5
2	—	—	—	—	—	—	—	1	1
Total	3	6	4	2	1	0	2	1	19

(*) In this case there is a shower in which secondary electron cascades confused mixedly the secondary p.p.s., and so n_s of the shower shows the minimum number of secondary p.p.s.

([19]) W. O. LOCK and G. YEKUTIELI: *Phil. Mag.*, **43**, 234 (1952).

([20]) U. CAMERINI, W. O. LOCK and D. H. PERKINS: *Progr. in Cosmic Ray Phys.*, vol. **1** (edited by J. G. WILSON, Amsterdam, 1952), p. 1.

TABLE V. – *Classification of S_t-showers according to n_s and N_h.*

N_h \ n_s	2	3	4	5	6	14	19	Total
0	—	2	—	—	—	—	—	2
1	1	—	1	1	1 (*)	1	—	5
2	1 (*)	—	1	1	—	—	—	3
3	—	—	—	—	—	—	—	0
4	—	2 (**)	1	—	1	—	—	4
5	—	—	—	—	—	—	—	0
6	—	1 (*)	1 (*)	—	1 (*)	—	—	3
7	—	—	—	—	—	—	1	1
Total	2	5	4	2	3	1	1	18

(*) In this case there is a shower in which secondary electron cascades confused mixedly the secondary p.p.s., and so n_s's of the shower shows the minimun number of secondary p.p.s.
(**) These two showers are those just described and n's show the minimum ones.

the results seriously. Also attention has to be paid to the fact that the observation has the tendency of underestimating the value of n_s, when secondary electron cascades are ejected to prevent the observation of secondary penetrating particles. The case is noted in the Tables. It is easily found that N_h of the P-shower is smaller than N_h of the S_t-shower. This fact is more definitely shown by average values of n_s and N_h ($\overline{n}_s$ and $\overline{N}_h$), which are given in Table VI. The P-shower and the S_t-shower have the same value of $\overline{n}_s$, 4.0 ± 0.5, wherein events of exceedingly great n_s are discarded. To the contrary the values of $\overline{N}_h$ are 0.3 and 2.8, respectively, for the P-shower and for the S_t-shower.

TABLE VI. – *Characteristics of secondary particles of P-showers and S_t-showers.*

	$\overline{n}_s$	$\overline{N}_h$	λ
P-shower	4.0 ± 0.5	0.3 ± 0.1	(26 ± 8) cm Pb
S_t-shower	4.0 ± 0.5	2.8 ± 0.4	(41 ± 14) cm Pb

The interaction mean free paths of secondary penetrating particles produced in the P-showers and in the S_t-showers are listed in Table VI, respectively. These values indicate the ratios of the number of the traversals of secondary penetrating particles through lead plates to that of nuclear interactions produced by them. In obtaining these data, we discarded pictures of penetrating showers in which secondary interactions have occurred in complex features. The discarding was made because, otherwise, we should have mis-counted the number of penetrations of secondary penetrating particles, but this should lead

to an overestimate of the m.f.p. In addition, most successive interactions were stars. If they occurred in a deep region within a lead plate, then they could be

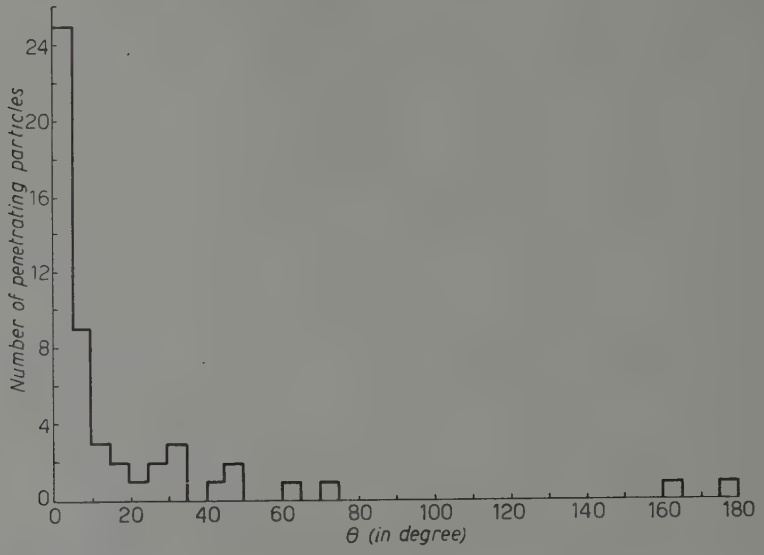

Fig. 3. – Distribution in projected angles, θ, between the incident and a secondary penetrating particle of the P-showers.

observed as if the incident penetrating particles had vanished without any coming out. One event of such a type was observed, but generally such an event would hardly be detectable, especially when it exists in the shower with secondary electron cascades. Hence, considering these circumstances, the interaction m.f.p. listed may have been overestimated, and may be consistent with the geometrical m.f.p. in lead, $\lambda_{\text{geom}} = 19.2$ cm Pb. That is, the secondary penetrating particles of the P-shower are considered to be the nucleonic component, probably π-mesons.

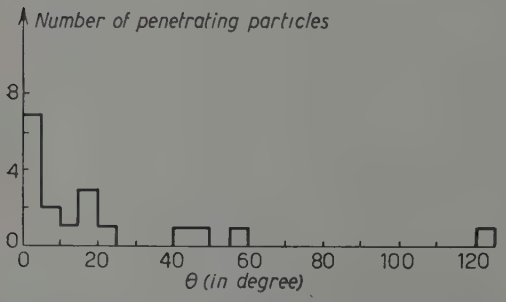

Fig. 4. – Distribution in projected angles, θ, between the incident and a secondary penetrating particle of the S_t-showers.

Distributions in projected angles of secondary penetrating particles against the direction of the incident particle are shown in Fig. 3 for the P-showers and in Fig. 4 for the S_t-showers (both are taken at 50 m w. e.). These figures seem to show a rather sharper

angular distribution of the P-showers than those of the S_t-showers, but the latter has large statistical errors, so that anything definite could not be concluded. Also distributions in projected angles of the secondary penetrating particles, which were ejected with minimum angles with respect to the incident direction and emerged without successive interaction, are shown in Fig. 5 for both types of showers. These data are also statistically insufficient, but not in disagreement with Kessler and and Maze's experiment ([16]). It is, however, to be mentioned that we have observed a few P-showers in which the secondary penetrating particles ejected with minimum angle underwent a successive interaction in the chamber.

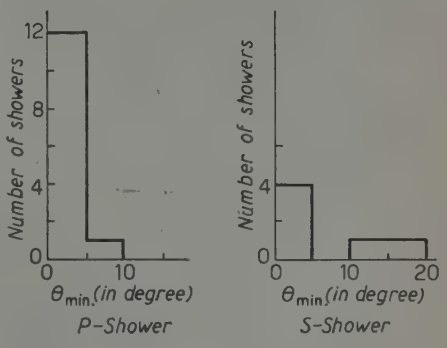

Fig. 5. – Distribution in projected angles, $\theta_{\min}$, between the incident and the minimum deflected secondary penetrating particle, which emerged from the cloud chamber without further interactions.

The average number of electron cascades is about 1.5 for both the P-shower and the S_t-shower taken at 50 m w. e. Almost all S_t-showers observed at 250 m w. e. have electron cascades of about 3 GeV energy or more. This fact may imply something to be clarified, but at present no conclusion is reached.

4. – Discussion and conclusions.

This experiment has established two remarkable facts, the appreciable contribution of the secondary nucleonic component to penetrating shower production underground, and scarce production of heavily ionizing secondaries of P-showers. The former is discussed with reference to the relation of depth-frequency of occurrence of penetrating showers and to the one expected by the Weizsäcker-Williams' theory, and the latter is discussed with reference to the mechanism of heavily ionizing secondary production.

4·1. *Contribution of the secondary nucleonic component to the penetrating shower production underground.* – The ratio of the number of penetrating showers produced by π-mesons to that of those produced by μ-mesons in the ground is experimentally deduced as follows. A penetrating shower produced by a π-meson is hereafter called a π-shower for brevity, the case being also for a μ-shower. Now it is reasonably assumed that all of the secondary particles of μ-showers are π-mesons. P-showers should be almost μ-showers. The pro-

bability of π-showers to be observed without an accompanying penetrating particle could be estimated after determining the angular distribution of secondary penetrating particles of μ-showers and considering the solid angle at the point of interaction subtended by the area of the cloud chamber. Because of the rather sharp angular distribution described above and the rather large observing area of the apparatus used, the contamination of π-showers into P-showers may be negligible in this experiment, this neglection giving the underestimate of the contribution of π-mesons to penetrating shower production underground.

On the other hand S_t-showers are entirely attributed to π-showers. Hence, from the experimental data listed in Table III, the ratio of the number of π-showers to that of μ-showers ($\equiv r$) is computed to be

$$r = \frac{\text{number of } \pi\text{-showers}}{\text{number of } \mu\text{-showers}} = \begin{cases} 1.14 \pm 0.40 \text{ at } 50 \text{ m w. e.}, \\ 0.83 \pm 0.35 \text{ at } 250 \text{ m w. e.}, \end{cases}$$

i.e., r is approximately equal to unity at both depths ([30]). It shows very clearly that the contribution of π-mesons cannot be neglected in the interpretation of underground experiments, not only on the star production but also on the penetrating shower production.

Consideration has to be given to the depth dependence of frequencies of P-showers, not including S_t-showers. As almost all P-showers are probably produced by μ-mesons, the frequencies of P-showers are what should be compared with a theory of μ-meson interaction such as the Weizsäcker-Williams' theory (*). The experimental results, listed in Table III, about the P-showers lead to the ratio of cross-sections at both depths as

$$\frac{\sigma(250 \text{ m w. e.})}{\sigma(50 \text{ m w. e.})} = 7.0 \pm 2.2 ,$$

while, in Table VII are listed the values predicted by the Weizsäcker-Williams' theory with constant γ-π cross-section for the assumed threshold energies ε. For the μ-meson nuclear interaction the threshold energy can not be determined without fundamental assumptions such as the similarity of features of μ-showers and π-showers. Hence the predicted values may be consistent with this experimental value within statistical error, if one can suppose the threshold energy ε of the μ-showers to be more than 15 GeV. Attention must be paid here to the fact that there is an appreciable contribution due to the

(*) The depth dependence of hodoscope data so far obtained has to be reinterpreted by taking into consideration the contribution of penetrating showers produced by nucleonic components. The reinterpretation will be given elsewhere.

transfer of a large portion of the incident energy to the predicted cross-sections as shown in Table VII.

TABLE VII. – *Predicted ratios of cross sections at* 250 m w.e. *and that of* 50 m w.e. *as a function of threshold energies* ε. *The cross sections refer to that of* μ*-meson induced showers and were evaluated by assuming the Weizsäcker-Williams formula with constant* γ-π *cross-section for the energy transfer spectrum in the* μ*-meson-nucleon interaction.*

ε (GeV)	3	5	7	10	20
$\frac{\sigma(250 \text{ m.w.e})}{\sigma(50 \text{ m w.e.})}$	2.9	3.5	4.0	4.8	7.1

Nevertheless, allowance must not be given in the interaction to such energy transfer.

4·2. *Scarce production of heavily ionizing secondaries in the P-shower.* – In this Section features of the *P*-shower are discussed in comparison with those of the S_t-shower. The experimental results show that the multiplicity, the angular distribution and the interaction m.f.p. of secondary penetrating particles of the *P*-shower are almost the same as those of the S_t-shower, which are tentative conclusions because of the rather large statistical error as described (*). Since the S_t-shower is considered to be produced probably by the π-meson, these results can be considered as consistent with such results hitherto obtained ([6,7,16]) which show that nuclear interactions underground have the same features as those of nuclear interactions observed at high altitudes. On the contrary, heavily ionizing secondaries were observed to be produced in the *P*-shower about ten times less frequently than in the S_t-shower, though about the same number of secondary penetrating particles were ejected in both showers. The experimental results are given in Table VI and Figs. 6, and 7. Discussion is

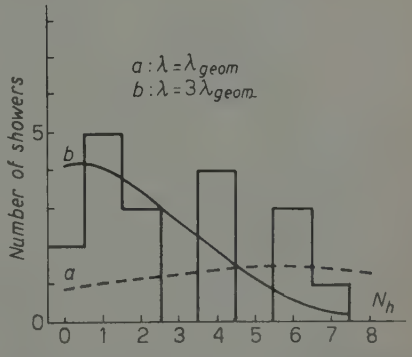

Fig. 6. – N_h-distribution of S_t-showers. The curve shows the evaluated $P_\pi(n)$ with $\lambda = \lambda_{\text{geom}}$ and $\lambda = 3\lambda_{\text{geom}}$, $P_\pi(n)$ being normalized to the value of 18. the total number of observed S_t-showers.

(*) The conclusion. deduced in the following, may not be changed, even when the angular distribution of secondaries will turn out to be sharper than that of π-showers.

made here especially of the scarce production of heavily ionizing secondaries in the P-shower.

First it has to be noticed that heavily ionizing secondaries, when observed in a cloud chamber by using lead plates 1 cm thick as in our experiment, correspond just to the grey tracks in nuclear emulsion. This is because the evaporation tracks of a star in emulsion are protons of energies less than 30 MeV, a proton of 30 MeV energy having the range 2 mm lead. This is also established as follows. From the data observed at high altitudes ([19,20]) we can get the energy spectrum of grey particles of showers to be $E^{-2.5}\mathrm{d}E$, and $\bar{N}_g=6$ as to be accompanied by showers of $n_s=4$ on the average. The reason why the shower of $n_s=4$ is picked up here is that the S_t-shower we observed had $\bar{n}_s=4$. Using these values and assuming uniform occurrence of showers in a lead plate of 1 cm thick, we could estimate the number of grey tracks coming out from the lead plate, and we found that the estimated value was in rough agreement with the experimental value 2.8 of the S_t-shower. Therefore heavily ionizing secondaries correspond to the grey tracks in nuclear emulsion, and in the following we will consider them as protons according to the results of the emulsion experiments ([19,20]). An attempt is made to interpret our experimental results on the basis of the hitherto observed facts; the interaction cross-section of high energy μ-mesons with matter is about 10^{-31} cm^2/nucleon, and the feature of secondary penetrating particles of its interaction is almost the same as those of penetrating showers observed at high altitudes.

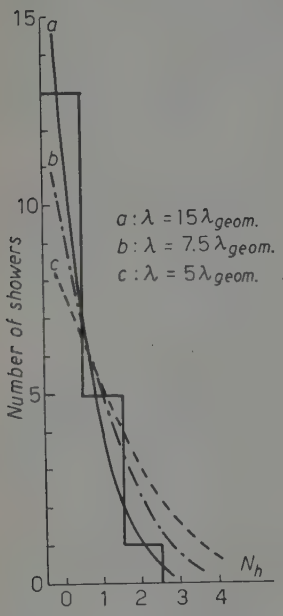

Fig. 7. – N_h-distribution of P-showers. The curves show the evaluated $P_\mu(n)$ with $\lambda=5\lambda_{\mathrm{geom}}$, $7.5\lambda_{\mathrm{geom}}$, $15\lambda_{\mathrm{geom}}$, respectively, $P_\mu(n)$ being normalized to the value of 19, the total number of observed P-showers.

Since heavily ionizing secondaries are protons emitted from the lead nucleus, they are probably not produced in the first encounter of the incident particle with a nucleon in the nucleus, but may come out through some conversion in the secondary or successive interaction in the very nucleus. In the problem considered, the difference of the case by μ-mesons and that by π-mesons is primarily represented by the position of the first encounter in the nucleus in consequence of the well established difference between their interaction cross-sections the cross-section of μ-mesons being about four order of magnitude smaller than that of π-mesons for lead nuclei.

Consider first the case that a π-meson went into a nucleus and took place

the first encounter with a nucleon in the nucleus. The target lead nucleus is reasonably approximated by the sphere of the continuous nuclear matter. If z denotes a length of residual portion, lying behind the position of the first encounter, in the nucleus, then the average of z is evaluated as follows,

$$\bar{z}_\pi = \frac{\int_0^R 2\pi y\,\mathrm{d}y \int_{-\sqrt{R^2-y^2}}^{+\sqrt{R^2-y^2}} \exp\left(-\frac{x+\sqrt{R^2-y^2}}{\lambda_\sigma}\right)(\sqrt{R^2-y^2}-x)\,\mathrm{d}x}{\int_0^R 2\pi y\,\mathrm{d}y \int_{-\sqrt{R^2-y^2}}^{+\sqrt{R^2-y^2}} \exp\left(-\frac{x+\sqrt{R^2-y^2}}{\lambda_\sigma}\right)\mathrm{d}x} \equiv \int_0^{2R} Z g_\pi(Z)\,\mathrm{d}z,$$

$$Z \equiv \sqrt{R^2-y^2}-x$$

where R is the radius of the target nucleus, $R = r_0 A^{\frac{1}{3}}$, and λ_σ is the interaction mean free path of π-mesons in a nucleus, being taken to correspond to the internuclear cross-section, 26 mb ([21,22]). A distribution function of the residual length z, $g_\pi(z)$, is also obtained in the course of the evaluation.

Next in the case of incidence of a μ-meson, $\bar{z}_\mu$ and $g_\mu(z)$ are obtained by making the λ_σ of the incident particle to infinity in the expression for the π-meson. Thus the followings were obtained

$$\bar{z}_\pi = 0.50 \cdot 2R = 1.87\,\lambda_\sigma$$

$$\bar{z}_\mu = 0.37 \cdot 2R$$

and $g_\pi(z)$ and $g_\mu(z)$ are shown in Fig. 8 (*).

([21]) M. Blau and A. R. Oliver: *Phys. Rev.*, **102**, 489 (1956).

([22]) L. Yuan and S. Lindenbaum: *Phys. Rev.*, **85**, 1827 (1952); **88**, 1017 (1952).

([23]) W. Imhof, E. A. Knapp, H. M. Watson and V. Perez-Mendez: *Phys. Rev.*, **108**, 1040 (1957).

(*) Attention has to be paid about $g_\mu(z)$. If the μ-meson is imagined to interact electromagnetically, its low energy interaction seems to be expected as occurring on the surface of a nucleus.

This is presumably based on the fact ([24]) that $A^{\frac{1}{3}}$ dependence has been obtained for the production cross section of a single π-meson of less than 120 MeV energy by bombarding 330 MeV or less γ-rays on various nucleus of different A's, in spite of great transparency as $\sigma_\gamma = 10^{-28}$ cm^2/nucleon. The experimental results were obtained in detecting π-mesons of a certain energy ejected at a certain angle against incident beams. At the present we have not the data of high energy interactions or multiple meson productions by γ-rays, so in the case of high energy interaction of μ-mesons like our experiment it is not evident whether its interactions occur on the surface of the nucleus or not.

However surface production of the γ-π interaction has reasonably been interpreted as characteristics of the low energy interaction ([24-26]) so that in the high energy inter-

([24]) K. A. Brückner, R. Serber and K. M. Watson: *Phys. Rev.*, **84**, 258 (1951).

([25]) S. T. Butler: *Phys. Rev.*, **87**, 1117 (1952).

([26]) R. R. Wilson: *Phys. Rev.*, **86**, 125 (1952); **104**, 218 (1956).

As it is easily seen from $g_\pi(z)$ and $g_\mu(z)$, the difference between occurring positions of the first encounters in both cases does not appear enough to be responsible for the difference of the value of N_h in the P-shower and that in the S_t-shower. Hence the production mechanism of grey secondaries, which is absent in the P-shower, is very interesting, but so far it has not been sufficiently investigated, especially in the case of such high energy that multiple meson production takes place. The only data have been provided by Bristol group ([19,20]) in the cosmic ray research. They investigated mainly the correlation of $\overline{N}_g$ and n_s in the multiple meson production in nuclear emulsions exposed at high altitudes, and concluded that about 80% of the number of grey particles are composed of protons, and that grey particles are partly due to the protons knocked out by penetrating particles traversing the very nucleus in which the penetrating particles were produced and partly due to the protons emitted after some conversion of energy of the first nucleon-nucleon or π-meson-nucleon encounter in the nucleus.

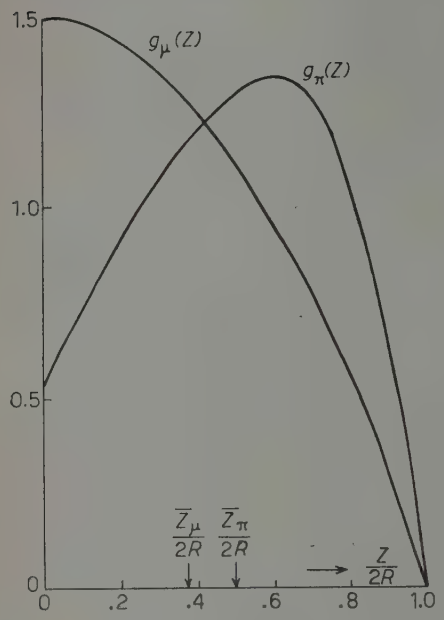

Fig. 8. – Residual length distribution, $g_\mu(z)$ and $g_\pi(z)$, of z, which is a distance in the nucleus lying behind the first interaction produced by μ-mesons and the nucleonic component (see the text).

Then we are interested in determining which mechanism of grey production is dominant. First is tested the assumption that the knock-on mechanism is dominant for the grey production in nuclear interaction of $\overline{n}_s=4$. If the assumption is correct, the difference between N_h of the P-shower and that of the S_t-shower may be reproduced according to the difference of $g_\mu(z)$ and $g_\pi(z)$. Let us assume that the grey particles are produced by knock-on mechanism and that there is no difference between secondaries of μ-meson-nucleon interactions and those of π-meson-nucleon interactions, the latter being hitherto recognized. The latter assumption is consistent with our experimental result that the inter-

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action it is expected not to be. While the A dependence of production cross-section of μ-showers has not been determined conclusively. In the case of this experiment the P-showers were selected by detecting the existence of two or more penetrating particles ejected at the shower origin, so we may naturally expect the occurrence of the interaction in the center region as well as on the surface of a nucleus.

action m.f.p.s. of secondaries both in the P-showers and in the S_t-showers were obtained to be consistent with the geometrical m.f.p. Since the number of collision of shower particles in the very nucleus is proportional to their pass length in the nucleus, the probability of occurring of n times collisions in the nucleus is evaluated as follows for the μ-shower by taking the statistical fluctuation into account,

$$P_\mu(n) = \int_0^{2R} g_\mu(z)\,\mathrm{d}z\,\frac{(m_z/\lambda)^n}{n!}\exp[-m_z/\lambda] =$$

$$= \frac{3}{2\beta^3}\left[\{\beta^2-(n+2)(n+1)\}\sum_{k=n+1}^{\infty}\frac{\beta^k}{k!}\exp[-\beta]+\beta(\beta+2+n)\frac{\beta^n\exp[-\beta]}{n!}\right],$$

where m is the multiplicity of secondary penetrating particles of the first encounter, λ the internuclear m.f.p., and $\beta = 2Rm/\lambda$. For the π-shower the following is similarly obtained,

$$P_\pi(n) = \left[\frac{\alpha^2}{2}-1+(\alpha+1)\exp[-\alpha]\right]^{-1}\cdot$$

$$\cdot\left[\frac{n+1}{m^2}\left\{\sum_{k=n+2}^{\infty}\frac{(m\alpha)^k}{k!}\exp[-m\alpha]\right\}+\frac{1}{m}\sum_{k=n+1}^{\infty}\left\{\frac{(m\alpha)^k}{k!}\exp[-m\alpha]\right\}-\right.$$

$$\left.-\frac{(\alpha+1)\exp[-\alpha]}{m-1}\left(\frac{m}{m-1}\right)\left\{\sum_{k=n+1}^{\infty}\frac{(m-1)^k\alpha^k}{k!}\exp[-(m-1)\alpha]\right\}\right],$$

where $\alpha = 2R/\lambda$. Assuming the one-to-one correspondence between the collision and grey production the experimental number distribution of grey prongs in the S_t-shower is consistent with the evaluated number of $P_\pi(n)$ using $\lambda = 2\lambda_{\mathrm{geom}}$ (see Fig. 6). This fact shows an approximate consistency between the experiment and the assumption we adopted, on the N_h of the S_t-shower, λ_{geom} corresponding to the internuclear cross-section of 26 mb. But the N_h distribution of the P-showers could not be explained by $P_\mu(n)$ with $\lambda = 2\lambda_{\mathrm{geom}}$, rather it is required to be at least $\lambda = 7\lambda_{\mathrm{geom}}$ for the completion of consistency (see Fig. 7). Taking into account the similarity of features of the secondaries of the P-shower and the S_t-shower about their multiplicity, angular distribution and interaction m.f.p., this discrepancy of N_h may indicate the incorrectness of the grey production mechanism we adopted. It is further ascertained as follows. The two values of λ, obtained above ($2\lambda_{\mathrm{geom}}$ and $\geqslant 7\lambda_{\mathrm{geom}}$), might be considered as the identical λ being due to statistical fluctuation. But this possibility is, with the probability consideration, denied on the significant level of 1% or less.

Thus, we could not explain the rare production of grey particles in the

P-shower on the basis of facts about μ-mesons hitherto obtained and the grey production mechanism. Therefore the experimental results suggest that the difference between the values of N_h of the P-shower and that of the S_t-shower has to be attributed to the difference between characteristics of elementary processes of μ-meson-nucleon collision and those of the π-meson-nucleon collision.

At present, it is not experimentally evident what mechanism is responsible for the difference. If the discussion, done by a part of us elsewhere, is accepted, the difference might be representing that slow mesons of about 200 MeV energies are produced much less frequently in π-meson-nucleon interactions than in π-meson-nucleon interactions, although both types of interactions containing the same number, 4, of shower particles on the average. This may contradict to the models of multiple meson production so far proposed ([27-29]), and so it is necessary to be further investigated.

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The authors wish to thank Professors M. Oda of the Tokyo University, S. Hayakawa of the Kyoto University and S. Miyake of Osaka City University for helpful discussions. They also wish to express their appreciation to the Shizuoka Railway Operating Divisional Office for permitting them to make the observation in the Isohama Tunnel. It is a pleasure to thank the Institute for Food Chemistry for financial aid.

([27]) S. Takagi: *Progr. Theor. Phys.*, **1**, 123 (1952).

([28]) W. L. Kraushaar and L. J. Marks: *Phys. Rev.*, **93**, 326 (1954).

([29]) H. W. Lewis: *Rev. Mod. Phys.*, **24**, 241 (1952); *Proc. of 7th Rochester Conf. on High Energy Nuclear Phys.*, XI-1.

RIASSUNTO (*)

Si usa, per osservare gli sciami penetranti sottoterra, una camera a nebbia a lastre multiple contenente quindici lastre di piombo dello spessore di 1 cm ciascuna. In 667.9 h ed a 50 m a.e. si ottengono quindici sciami penetranti in 3603.1 h, ed a 250 m a.e. si ottengono ventitrè sciami penetranti, aventi, rispettivamente, in media,

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(*) *Traduzione a cura della Redazione.*

quattro particelle secondarie. Poichè la camera, di grande larghezza (100 cm), è stata posta quanto più vicino possibile alla parete superiore del tunnel, si rivolge particolare attenzione nel distinguere gli sciami penetranti prodotti dai mesoni μ da quelli prodotti dalla componente nucleonica. Quasi tutti gli sciami generati da particelle isolate incidenti (chiamati fenomenologicamente sciami P) si considerano come prodotti con ogni probabilità dai mesoni μ, mentre si riguardano come dovuti alla componente nucleonica quegli sciami (chiamati sciami S) generati da una sola delle due o più particelle incidenti, ciò per il fatto che il cammino libero medio della componente nucleonica per le interazioni nucleari, è di circa 10^{-4} volte più breve di quello dei mesoni μ. Eseguita la correzione per il rendimento dell'apparecchio, si trova che i rapporti fra le frequenze degli sciami S e degli sciami P sono 1.1 ± 0.3 e 0.92 ± 0.23 ad entrambe le profondità, la qual cosa significa che metà delle interazioni nucleari sotterranee di alta energia è prodotta dalla componente nucleonica. Si confronta la dipendenza delle frequenze degli sciami P dalla profondità, con le previsioni della trattazione di Weizsäcker e Williams sulle interazioni dei mesoni μ. Si è osservato inoltre, che gli sciami P hanno, in modo rimarchevole, caratteristiche diverse da quelle degli sciami S, cioè il numero medio di secondari fortemente ionizzati degli sciami P è 0.3 per sciame, mentre il valore per gli sciami S è 2.8 per sciame.

Decay Rate and Spectrum of Electrons from μ^--Mesons of the K-Shell.

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(ricevuto il 20 Marzo 1959)

Summary. — The decay rate of a μ^- meson from the K-shell of a light nucleus is evaluated as a function of the momentum of the produced electron and of the atomic number, taking into account the main electrostatic corrections on the usual, plane wave eigenfunctions of the electron, by a Born-like approximation. Such corrections change appreciably the decay rate and the shape of the spectrum (mainly, its lowest part) of the emitted electrons.

1. – Introduction.

Wheeler and Tiomno first [1] calculated the decay rate of a μ^-, captured in the K-shell of a nucleus, in an electron and a neutrino-antineutrino pair. The calculation of these authors concerns mainly the ratio of the probability of decay to the probability of absorption of the meson by a proton, and is rather crude in so far as they take into account very roughly the true space-distribution of the μ^- and neglect completely the electrostatic interactions of the electron with the atom in which the decay takes place. More accurate calculations have been done by [2] Porter and Primakoff and by Muto [3] *et al.*, but they too neglect the electrostatic actions on the electron. The decay rate given by Muto is not convenient, above all, to get a figure, because of the complicate mixing of different kinds of interactions.

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[1] J. Tiomno and J. Wheeler: *Rev. Mod. Phys.*, **21** 153, 133 (1949)
[2] C. Porter and H. A. Primakoff: *Phys. Rev.*, **83**, 849 (1951).
[3] T. Muto, M. Tanifuji and K. T. Inoue: *Progr. Theor. Phys.*, **8**, 13 (1952).

In the following an evaluation of the decay probability of the μ^- meson is reported taking into account in first approximation the electrostatic actions on the electron, which contribute (modifying the unperturbed plane wave eigenfunctions of the emitted particle) in a decisive manner to the overall decay rate, and mainly to the lowest part of the spectrum of the electrons. Mention is also made of the contributions from higher order approximation terms.

From a general point of view it is evident that a precise evaluation of the decay rate allows, first of all, to fix with accuracy the coupling constants of the weak interactions between muon, electron and neutrino-antineutrino: indeed, even for most of the light nuclei, the nuclear recoil is negligible and therefore K-shell decay can be preferred to the free muon decay. On the other hand, absorption of K-shell mesons from the nuclear protons turns out to be a function of the proton density and, more important still, of the nuclear excitation, which depends in a complicate way on the nuclear structure. For instance, excited unstable nuclei can be produced in the absorption, as [4] GODFREY and JACKSON, TREIMAN and WYLD pointed out. The dynamics of such a process is a very complicate one and it looks evident that a better understanding of the direct decay probability, associated to the experimental data on the decay/absorption probability ratio, becomes very useful for the interpretation of the nuclear structure (probably better than other effects, such as scattering of particles and photons). From this point of view, the experiments performed recently by [5] TELEGDI *et al.* on the decay rate of μ^-'s in nuclei of different atomic number Z are very interesting.

This quantity depends on Z owing to the extension of the μ^- eigenfunctions, to the electrostatic actions of the atom on the produced electron, and to the available energy: the latter is a function of the binding energy of the μ^-. For high Z a strong correction to the nuclear extension, is caused by the above quantities. Relativistic corrections, as well, have to be taken into account (particularly for vacuum polarization) also on muonic and electronic charges. However for light nuclei such effects are negligible and the following evaluation shall, therefore, disregard them.

2. – Particles eigenfunctions.

To write down the matrix elements for the decay interaction

$$\mu^- \to e^- + \nu + \bar{\nu} \tag{1}$$

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[4] J. D. JACKSON, S. B. TREIMAN and H. W. WYLD: *Phys. Rev.*, **107**, 327 (1957); S. B. TREIMAN: *Phys. Rev.*, **410**, 448 (1958).

[5] R. A. LUNDY, J. C. SENS, R. A. SWANSON, V. L. TELEGDI and D. D. YOVANOVITCH: *Phys. Rev. Lett.*, **1**, 102 (1958).

it is convenient to choose as representation of the Dirac matrices, the one used by (6) YENNIE, RAVENHALL and WILSON in the problem of high energy electron scattering, and (7) by LEE and YANG for β decay. In such a representation, the equation of motion of an electron in the electrostatic field of potential V is written for the eigenfunctions

$$\psi_e(\boldsymbol{r}) = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \qquad \begin{aligned} (H - \boldsymbol{\sigma}\cdot\boldsymbol{p})\varphi &= -V\varphi + m\chi \\ (H + \boldsymbol{\sigma}\cdot\boldsymbol{p})\chi &= -V\chi + m\varphi \end{aligned} \tag{2}$$

(we put $\hbar = e = c = 1$; m is the electron mass). φ and χ are opposite spin spinors; losely speaking, if $m \ll H$, φ and χ are (5) eigenfunctions with $\boldsymbol{\sigma}$, respectively, parallel and antiparallel to $\boldsymbol{p}$. Spinors φ and χ are connected to the corresponding φ' and χ' of the Pauli representation for Dirac matrices, by the unitary transform

$$\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = 2^{-\frac{1}{2}} \begin{pmatrix} \chi' - \varphi' \\ \chi' + \varphi' \end{pmatrix} \tag{3}$$

and then, in the Y.R.W. representation, large and small components are mixed. As far as the eigenfunctions of the electrons subject to the nuclear electrostatic potential are concerned, if we introduce in the matrix element the exact eigenfunctions, the calculation turns out almost impossible. Considering that electrons have mainly high energy, it is enough to use for their eigenfunctions a particular, Born-like approximation. In the following we want to have momentum eigenfunctions: if we put

$$\begin{aligned} \varphi(\boldsymbol{p}) &= \varphi_0 + \varphi_1(\boldsymbol{p}) \\ \chi(\boldsymbol{p}) &= \chi_0 + \chi_1(\boldsymbol{p}) \end{aligned} \qquad \varphi_0 = \chi_0 = \delta(\boldsymbol{p} - \boldsymbol{k}) \tag{4}$$

in which φ_0, χ_0 are eigenfunctions of the homogeneous ($V = 0$) Dirac equation (2) and if

$$\Delta^+ = E_p + \boldsymbol{p}\cdot\boldsymbol{\sigma}\,, \qquad \Delta^-(\boldsymbol{p}) = E_p - \boldsymbol{p}\cdot\boldsymbol{\sigma}\,, \tag{4'}$$

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(6) D. R. YENNIE, D. G. RAVEHALL and R. N. WILSON: *Phys. Rev.*, **95**, 500 (1954), in the following cited as Y. R. W. ALSO, J. RAINWATER: *Ann. Rev. Nucl. Sci.*, **7**, 1 (1957).

(7) C. N. YANG and T. D. LEE: *Phys. Rev.*, **105**, 1671 (1957); also RAINATER (6).

we have

$$(5)\quad \begin{cases} \varphi_1(\boldsymbol{p}) = -\dfrac{\Lambda^+(\boldsymbol{p})}{2m^2}\int V(-\boldsymbol{k})[\varphi_0+\varphi_1(\boldsymbol{p}+\boldsymbol{k})]\,\mathrm{d}\boldsymbol{k} - \dfrac{1}{2m}\cdot \\ \qquad\qquad \cdot\int V(-\boldsymbol{k})[\chi_0+\chi_1(\boldsymbol{p}+\boldsymbol{k})]\,\mathrm{d}\boldsymbol{k}\,, \\ \chi_1(\boldsymbol{p}) = -\dfrac{\Lambda^-(\boldsymbol{p})}{2m^2}\int V(-\boldsymbol{k})[\chi_0+\chi_1(\boldsymbol{p}+\boldsymbol{k})]\,\mathrm{d}\boldsymbol{k} - \dfrac{1}{2m}\cdot \\ \qquad\qquad \cdot\int V(-\boldsymbol{k})[\varphi_0+\varphi_1(\boldsymbol{p}+\boldsymbol{k})]\,\mathrm{d}\boldsymbol{k}\,. \end{cases}$$

Equation (5) can be solved with an iterative method, as every integral equation. In practice, if the electrostatic action is small enough, we can stop the development at the lower powers of the electric charge. In the following we use as approximated electron eigenfunction, the function

$$(6)\qquad \psi_e(\boldsymbol{p}) = N_k \begin{pmatrix} \delta(\boldsymbol{p}-\boldsymbol{k}) - V(-\boldsymbol{p})/2m[1+\Lambda^+(\boldsymbol{p})/m] \\ \delta(\boldsymbol{p}-\boldsymbol{k}) - V(-\boldsymbol{k})/2m[1+\Lambda^-(\boldsymbol{k})/m] \end{pmatrix},$$

where N_p is a normalization factor: with the normalization of an electron per momentum interval

$$\int |N_p|^2[\varphi^+\varphi + \chi^+\chi]\,\delta(\boldsymbol{p}-\boldsymbol{k})\,\mathrm{d}\boldsymbol{k} = 1\,,$$

N_p turns out nearly unitary.

To fix the μ^- eigenfunctions in momentum space, care must be taken that the radial eigenfunction in configuration space is singular at the origin (for $r=0$, $[1-\alpha^2Z^2]-1$ singularity). But near this point, owing to the finite extension of the nucleus, the electrostatic potential is not singular: let us start from the usual oscillator model of the nucleus (well approximated [7] for the lowest values of Z). In such a case the electrostatic potential of the nucleus is given by

$$(7)\qquad V(r) = \frac{2Z}{\sqrt{\pi}}\,\frac{\mathrm{Erf}\,(\omega r)}{\omega r} + O\,\exp\,[-r^2\omega^2]\,,$$

if ω is the harmonic potential strength, (for unitary mass), Erf (z) is the Gauss error function and O is a function nearly zero for $r\sim\omega$. It is evident that for $r\sim\omega$ a correction must be introduced in the μ^- eigenfunction: such a correction is well approximated by a function different from zero in the range

$$I:\quad 0\leqslant r\leqslant n\omega\,, \qquad\qquad \omega\sim 1,$$

to which we adjust the usual radial eigenfunction. But the range I is of the same order of magnitude of the interval in which the radial eigenfunction differs appreciably from the non-relativistic eigenfunction: so we can substitute to the relativistic, the Schrödinger radial eigenfunction: other relativistic constant corrections are negligibly small. We have with a good accuracy, in the Y.R.W. representation

$$\psi_\mu(\boldsymbol{P}) = 2^{-\frac{1}{2}}\left(1+\frac{2}{3}\alpha^2Z^2\right)\frac{16\pi a_0^{\frac{5}{2}}}{[a_0^2+P^2]^2}\begin{pmatrix} Y_{00}+\lambda P Y_{10} \\ \lambda\sqrt{2}\, Y_{11}P \\ -Y_{00}+\lambda P Y_{10} \\ \lambda\sqrt{2}\, P Y_{11}\end{pmatrix} \tag{8}$$

where a_0 indicates the Bohr momentum of the meson,

$$\lambda = \frac{\alpha Z}{2a_0 3^{\frac{1}{2}}} \qquad (\alpha = \text{fine structure constant})$$

and $\boldsymbol{P}$ is the muon's momentum, Y_{00}, etc., are functions of $\boldsymbol{P}/P$.

3. – Decay rate.

If

$$\mathcal{H}_{\text{in}}(\boldsymbol{x}) = g_i\psi_e^+(\boldsymbol{x})O_i\psi_\mu(\boldsymbol{x})\cdot\psi_\nu^+O_i\psi_{\bar\nu}(\boldsymbol{x}) \tag{9}$$

is the interaction hamiltonian density for the decay process (1), and if we assume plane wave eigenfunctions

$$\psi_\nu(\boldsymbol{x}) = \varphi_\nu\exp[i\boldsymbol{k}\cdot\boldsymbol{x}]\,, \qquad \psi_{\bar\nu} = \varphi_{\bar\nu}\exp[-i\boldsymbol{k}'\cdot\boldsymbol{x}] \tag{10}$$

for the neutrino-antineutrino field, the decay probability per unit time is given by

$$\mathcal{P} = (2\pi)^{-5}\int \mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}'\,|(f|M|i)|^2\delta(E_\mu - E_e - K - K')\,. \tag{11}$$

In eq. (11), $(f|M|i)$ is the matrix element

$$g_i\int \mathrm{d}\boldsymbol{P}\,\mathrm{d}\boldsymbol{p}\,\psi_e^+(\boldsymbol{p})O_i\psi_\mu(\boldsymbol{P})\varphi_\nu^+O_i\varphi_{\bar\nu}^-\;\delta(\boldsymbol{P}+\boldsymbol{p}-\boldsymbol{k}-\boldsymbol{k}') \tag{11'}$$

K, K' are the neutrino and antineutrino's energies. We note that in eq. (11) the usual symbol of average on the spin states has not been put, because neutrino

and antineutrino are longitudinally polarized: hence, integration on all $\boldsymbol{k}$ and $\boldsymbol{k}'$ means sum over all the possible spin directions. For the μ^-'s and e^-'s eigenfunctions (6) and (8) are already a combination of the possible spin states.

To simplify the calculation of $\mathcal{P}$, let us introduce, instead of $\boldsymbol{k}$, $\boldsymbol{k}'$, the two vectors

$$\boldsymbol{a} = \boldsymbol{K} + \boldsymbol{K}', \quad \boldsymbol{\alpha} = \boldsymbol{k} + \boldsymbol{k}', \qquad \boldsymbol{b} = \boldsymbol{k} - \boldsymbol{k}' ;$$

momentum (and energy) conservation is directly expressed in $(a, \boldsymbol{a})$, while $\boldsymbol{b}$ is a vector whose modulus varies between zero and a.

The electrostatic potential acting on the electron is that of a point nucleus (Ze) shielded by external electrons

$$V(r) = \frac{Z \exp[-cr]}{r}, \qquad c = 0.183\,7 Z^{\frac{1}{3}}(a_1)^{-1}, \tag{12}$$

where a_1 is the Bohr radius for the H atom ([9]). At this point it is convenient to express in $\mathcal{P}$, the energies and momenta in electronic mass units. If we put in such a system

α = $Z \times$ fine structure constant

$a_0 = \mu \cdot \alpha$ $\qquad$ μ = ratio of μ^- meson to electron mass

$W = \mu - 2a_0 - \sqrt{p^2 + 1}$ $\qquad$ $\boldsymbol{p}$ = electron's momentum

$\lambda = (12\mu^2)^{-1}$ $\qquad$ $c = 0.183\,7/\mu Z^{\frac{1}{3}}$

$X_\pm = [a_0^2 + (W \pm p)^2]^{-1}$

the decay probability $\mathcal{P}$ in the used approximation is given by

$$\mathcal{P} = \frac{2}{3\pi} |g_i|^2 a_0^5 W^3 \left\{ \frac{6W}{p} [(1 - 3\lambda^2)(X_-^3 - X_+^3) + 2\lambda^2 (X_-^2 - X_+^2)] + \right. \tag{13}$$

$$+ \frac{1}{32\pi^2} \frac{W^2}{\mu^2} \left[\frac{1}{W^2 + (a_0 + c)^2} + \frac{A}{2W} \right]^2 - \frac{W}{4p\pi^2\mu} (X_- - X_+) \left[\frac{1}{W^2 + (a_0 + c)^2} + \frac{A}{2W} \right] +$$

$$\left. + \frac{1}{(16\pi\mu)^2} \frac{W^4}{[W^2 + (a_0 + c)^2]^2} \right\} \cdot N_p^2 \left(1 + \frac{2}{3}\alpha^2 \right)^2 \mathrm{Spur} |O_i|^2 .$$

A is a rather complicated function of W, a_0, etc. (see Appendix).

([8]) For example: M. G. Mayer and J. H. Jensen: *Elementary Theory of Nuclear Shell Structure* (New York, 1955), chap. 4; I. Talmi: *Helv. Phys. Acta*, **25**, 185 (1952).

([9]) For example: P. Gombas: *Handb. d. Phys.*, vol. 36 (Berlin 1956), p. 157 (with a small difference).

4. – Conclusions.

A noticeable result of the preceding evaluation is the strong contribution to $\mathcal{P}$ from the terms of the e^- eigenfunction, which derives from the electrostatic perturbation on the plane wave functions; this happens in parti-

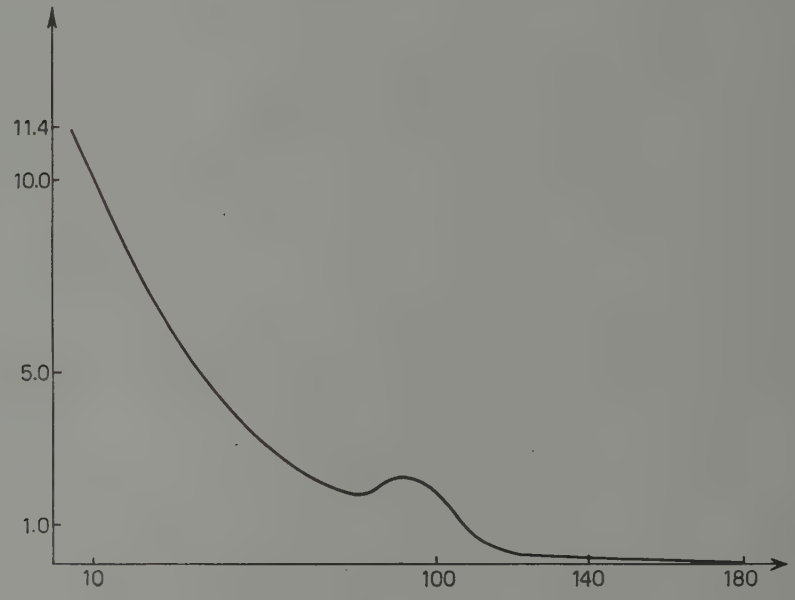

Fig. 1.

cular for the last term in brackets, which contributes strongly to the overall emission, increasing mainly the lowest part of the spectrum (large values of the available energy W). Such a fact is clearly shown by Fig. 1 in which the function $F(p)$ is reported for $Z = 6$:

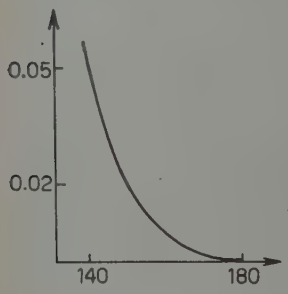

Fig. 1*a*.

$$\mathcal{P}(p) = \frac{2}{3\pi} a_0^5 |g_i|^2 \,\mathrm{Spur}\, |O_i|^2 F(p) \,.$$

With the plane wave e^- eigenfunctions we should obtain a $F(p)$ which vanishes for the lowest values of p (as is evident from eq. (13)) and has a maximum at $p = 90 \div 100$.

But, the amount of increase of $\mathcal{P}(p)$ due to the electrostatic correction on $\psi_e^0(\boldsymbol{p})$ could raise the doubt that higher order corrections on ψ_e^0 destroy the

preceding results. We have performed a rough extimation for second order correction on $\psi_e^0(\boldsymbol{p})$ (from formula (5) and for $Z = 6$) based on the evaluation of the main terms. We deduce that $\int \mathcal{P} p^2 \, dp$ decreases, but the correction is very small and negligible also for the lowest p values.

$\mathcal{P}(p)$ is a function of Z: besides the a_0^5 dependence, it turns out to be a decreasing function of Z as is illustrated by Fig. 2 in which we report the picture of

$$10^{-3} \int F(p) p^2 \, dp \, ,$$

for a set of values of Z.

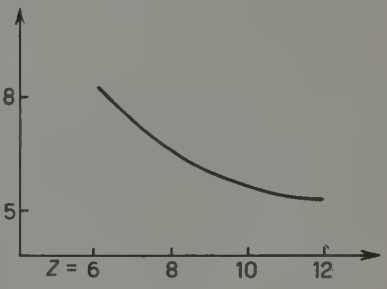

Fig. 2.

From the preceding considerations it is evident that formula (13) is a well approximated one for $Z \leqslant 15$: for heavier and more extended nuclei, strong deviations originate from the corrections on the μ^- eigenfunction.

Appendix

The factor A in eq. (13) derives from

$$\frac{1}{(2\pi)^3} \int \frac{d\boldsymbol{P}}{(a_0^2 + P^2)^2} \frac{E(\boldsymbol{P} - \boldsymbol{a})}{2m} V(\boldsymbol{P} - \boldsymbol{a}) \, .$$

If we write

$$r = [m^4 + 2m^2(W^2 - a_0^2) + (a_0^2 + W^2)^2]^{\frac{1}{2}} \, ,$$

$$\theta = \frac{1}{2} \operatorname{arctg} \frac{2 W a_0}{W^2 + m^2 - a_0^2} \, , \qquad b^2 = m^2 - c^2 ,$$

it is

$$A = r^{-\frac{1}{2}} \{ W \cos\theta + a_0 \sin\theta + b^2 (b^2 + r)^2 - 4 r b^2 \cos^2\theta)^{-1} \cdot$$
$$\cdot [(2r \cos\theta - r - b^2)(W \cos\theta + a_0 \sin\theta) - r \sin\theta (W \sin\theta - a_0 \cos\theta)] \} .$$

RIASSUNTO

Si è calcolata la probabilità di decadimento di un mesone μ^-, legato nel K-shell d'un nucleo leggero, in funzione del momento dell'elettrone di decadimento. Le correzioni dovute all'interazione elettrostatica fra elettrone e nucleo modificano sensibilmente la probabilità di decadimento e, soprattutto, lo spettro degli elettroni emessi.

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Note on the Conserved Current in the Weak Interactions (*).

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(ricevuto il 24 Marzo 1959)

Summary. — The relation between the lack of renormalization and the conserved current is investigated. Furthermore, it is proved under some conditions that we cannot construct a conserved current for the strangeness-violating process. However, if we neglect the mass-differences between baryons, then this is possible to do and some examples are given. Finally, some speculations on the weak interactions are given.

1. – Introduction.

Recently, the $V-A$ theory (1) of the weak interactions has been well established, as far as the strangeness-conserving processes are concerned. One interesting point is an apparent lack of renormalization for the vector part of the β-decay matrix element. This led some authors (2) to the concept of the conserved vector current. However, in the case of the axial vector current, the assumption of a conserved current gives too large an effective pseudoscalar term (3), which contradicts experiment. Furthermore, the electron decay of the pion would be forbidden (4). Thus, in this case, the idea of a conserved current does not seem to work at all.

(*) Work supported in part by the U. S. Atomic Energy Commission.

(1) E. C. Sudarshan and R. E. Marshak: *Proceedings of the Padua-Venice Conference* (September 1957); *Phys. Rev.*, **109**, 1860 (1958); R. P. Feynman and M. Gell-Mann: *Phys. Rev.*, **109**, 193 (1958); J. J. Sakurai: *Nuovo Cimento*, **7**, 649 (1958).

(2) R. P. Feynman and M. Gell-Mann: See reference (1).

(3) M. L. Goldberger and S. B. Treiman: *Phys. Rev.*, **110**, 1478 (1958).

(4) J. C. Taylor: *Phys. Rev.*, **110**, 1216 (1958).

For the strangeness-violating processes, the V-A theory is not so well established. However, the very small probability for K_{e_2} mode, compared to K_{μ_2} mode may support the idea of the universal V-A theory, since the probability for K_{e_3} mode is almost equal to that for K_{μ_3} decay. Hereafter, we assume the universal V-A theory even for the strangeness-violating processes. Then, we can ask a similar question whether the current in this case is conserved or not. If the axial vector part were conserved, the K_{μ_2} decay as well as K_{e_2} decay would be forbidden (*). For the case of the vector part, it is still an open question whether there is a conservation law. If this current were conserved, then we would predict that the electron decay of the positive kaon is more probable than the corresponding muon decay ([3]) which seems to be consistent with the experiment ([5]). Furthermore, the angular distribution for these decays would be uniquely determined ([6]), which may be tested experimentally.

In this note, first we shall show that in the case of the strangeness-violating current, it is actually not possible to construct a conserved current under certain conditions. However, if one can neglect the mass differences between the nucleon and the hyperons, it is possible to do so and some of the examples of a conserved vector current are given in the Section **4**. These approximately conserved currents have some interesting features, apart from the conservation itself, and some speculations concerning these aspects will be given.

Secondly, we shall prove that when the current is conserved, there is no need to renormalize under certain conditions. Although this proof may not be interesting from a practical standpoint, it may be illuminating to find the relation between the conservation of the current and the lack of renormalization, which is a certain generalization of Ward's identity ([7]).

2. – Conservation of the strangeness-violating current.

Here, we shall give an argument against the possibility of constructing a conserved current j_μ of the usual type for the strangeness-violating process.

Suppose that we have

$$\frac{\partial}{\partial x_\mu} j_\mu(x) = 0 \,. \tag{1}$$

(*) *Note added in proof.* - We assumed that K-meson is pseudoscalar.

([5]) See, *e.g.* M. Gell-Mann and A. H. Rosenfeld: *Ann. Rev. Nucl. Sci.*, **7**, 407 (1957).

([6]) S. Weinberg, R. E. Marshak, S. Okubo, E. C. G. Sudarshan and W. B. Teutsch: *Phys. Rev. Lett.*, **1**, 25 (1958).

([7]) J. Ward: *Phys. Rev.*, **78**, 182 (1950).

Let us construct the U; the charge operator of this current

$$U = \int \mathrm{d}^3 x j_4(x, t) . \tag{2}$$

Then, by virtue of Eq. (1), U is independent of the time t, and therefore we have

$$[H, U] = 0 ,$$

where H is the total Hamiltonian of the system. When Ψ is an eigenstate of H with the eigenvalue E, namely when

$$H\Psi = E\Psi \tag{4}$$

then, Eq. (3) gives

$$H(U\Psi) = E(U\Psi) . \tag{5}$$

Now, take for Ψ the one nucleon state. In that case, Eq. (5) implies that $U\Psi$ must again be the one nucleon state, since there is no other state which has the same energy (*). But this contradicts the assumption that j_μ is a strangeness-violating current. From this, we may conclude that we cannot construct a conserved strangeness-violating current. Of course, if we neglect the mass-difference between the nucleon and the hyperon, this conclusion is not necessarily true, because in that case, the one hyperon state will have the same energy as the nucleon. Thus, the conservation of current must necessarily be an approximate one made possible by the neglect of the mass-difference. Explicit examples of this will be given in the Section **4**.

However, this argument given in the above has one drawback; if $U\Psi$ is identically zero, the argument fails. We consider this possibility to be unlikely, and in fact if j_μ has the usual form Eq. (6) we can show that $U\Psi \neq 0$.

$$j_\mu = f(\overline{Y}\gamma_\mu Q N) + F_\mu . \tag{6}$$

In Eq. (6), Y and N stand for any hyperon, (Λ or Σ) and nucleon field operators, respectively, and Q for a numerical matrix, and F_μ may consist of meson variables (pion and kaon) and its derivatives. Furthermore we assume Eq. (7).

$$[Y(x), F_4(x')] = 0 \qquad (x_0 = x_0') . \tag{7}$$

(*) Here, we have assumed that j_μ conserves the baryon number and we neglect the weak interactions in H.

Eq. (7) will be satisfied if the interaction Hamiltonian does not contain any derivative interactions and if F_μ does not contain any baryon operators, namely if F_μ consists entirely of meson variables. Now, we prove that we cannot have

$$U\Psi_N = 0 . \tag{8}$$

By an argument similar to that given after Eq. (5), we must have also the following equation for the vacuum state Ψ_0, if the current is conserved

$$U\Psi_0 = 0 , \qquad U^+\Psi_0 = 0 . \tag{9}$$

Thus, if Eq. (8) and Eq. (9) hold, then for an arbitrary operator A we must have

$$(\Psi_0^*, [U, A]\Psi_N) \equiv 0 . \tag{10}$$

Now, take A to be the one hyperon operator $Y(x)$. Then by virtue of Equations (2), (6) and (7), we must have

$$-f(\Psi_0^*, Q\,N(x)\Psi_N) = 0 .$$

This is not possible, however, since

$$(\Psi_0^*, N(x)\Psi_N) = V^{-\frac{1}{2}} Z_2^{\frac{1}{2}} \exp[ipx]\, u_N(p) \neq 0 , \tag{11}$$

where V is the normalization volume, Z_2 is the renormalization constant, and $u_N(p)$ is the Dirac spinor for the nucleon. Thus, we conclude that Eq. (8) does not hold for the current of Eq. (6) (*).

Summarizing results in this section, we find that we cannot construct a conserved strangeness-violating current j_μ of Eq. (6) under the condition Eq. (7). If j_μ does not satisfy these conditions, we cannot say much beyond the fact that it must satisfy Eq. (8) in order to have the conservative law Eq. (10). For any special form of an j_μ, we may check this by means of Eq. (10). Furthermore, by using a similar method, we see that we must have

$$\begin{cases} U^+\Psi_N = 0 , & U\Psi_N = 0 , \\ U^+\Psi_\pi = 0 , & U\Psi_\pi = 0 , \\ U^+\Psi_{N+\pi} = 0 , & U\Psi_{N+\pi} = 0 , \end{cases} \tag{12}$$

in order that Eq. (1) holds. The last equation in Eq. (12) is true only for a $\Psi_{N+\pi}$ state such that its total energy is different from that of a Λ-hyperon, and less

(*) *Note added in proof.* - B. Sakita informed the author that current $j_\mu = \partial_\nu(\bar{\Lambda}\sigma_{\mu\nu}p)$ is conserved. However, U is identically zero ($U \equiv 0$).

than the sum of the rest-energies of Λ and pion. It is quite unlikely that all of Eq. (12) can be satisfied, though we have proved the impossibility only for the special form of Eq. (6).

3. – Lack of renormalization.

Here, we prove the lack of renormalization when the current j_μ is conserved. For simplicity, we restrict ourselves to the case of the nucleon current and furthermore, we assume that j_μ has the following structure:

$$j_\mu = f\bar{N}\gamma_\mu Q N(x) + F_\mu , \tag{13}$$

where $N(x)$ is the unrenormalized nucleon field operator and Q is a matrix constructed from the γ and τ matrices (*e.g.* $Q = \gamma_5 \tau_+$). Moreover, we assume

$$[N(x), F_4(x')] = 0 \qquad (x_0 = x_0'), \tag{14}$$

Eq. (14) will be satisfied, if F_μ consists entirely of the meson variables $\varphi(x)$ and $(\partial/\partial x_\mu)\varphi(x)$ and if the interaction Hamiltonian is non-derivative *i.e.* ps-ps but not ps-pv. Eq. (13) and Eq. (14) correspond to Eq. (6) and Eq. (7) of the previous section. Furthermore, we construct the operator U by Eq. (2) again.

Now, we have four nucleon states with two isotopic and two spatial spin components. We specify them by Ψ_α $(\alpha = 1, 2, 3, 4)$.

From the argument following just after Eq. (5), and the conservation of the nucleon number in U we must have

$$U\Psi_\alpha = \sum_\beta a_{\beta\alpha}\Psi_\beta \qquad (\alpha = 1, 2, 3, 4), \tag{15}$$

where the $a_{\beta\alpha}$ are constants and the summation on β runs from $\beta = 1$ to $\beta = 4$. From Eq. (15),

$$a_{\beta\alpha} = (\Psi_\beta^* U \Psi_\alpha) . \tag{16}$$

On the other hand, from Equations (2), (13) and (14), we have

$$[N(x), U] = fQN(x) .$$

Taking the matrix element between the vacuum and the one nucleon state, we have

$$\sum_\beta a_{\beta\alpha}(\Psi_0^* N(x)\Psi_\beta) = f(\Psi_0^* Q N(x)\Psi_\alpha) , \tag{17}$$

where we have set

$$U^+\Psi_0 = 0 \,. \tag{18}$$

In this case, it is necessary to explain Eq. (18). Actually, by the same argument which led to Eq. (15), we get

$$U^+\Psi_0 = a_0\Psi_0 \,,$$

where a_0 is a constant. It turns out that in most cases $a_0 = 0$, and we get Eq. (18). For example, if Q contains the τ-spin as in the case of β-decay, or if Q contains a γ_5 matrix, then by charge independence, or by parity conservation, a_0 must be zero. In other cases, if from the beginning we take the normal product for j_μ in Eq. (13), we generally again have $a_0 = 0$. If a_0 is still not zero, we can subtract a_0 from U at the start, which implies a change of definition of F_μ by an additive constant.

Now, by Eq. (11),

$$(\Psi_0^* N(x)\Psi_\alpha) = V^{-\frac{1}{2}} \cdot Z_2^{\frac{1}{2}} \exp[ipx] u_\alpha(p) \,, \tag{11}$$

where u_α is normalized to

$$u_\alpha^*(p)\, u_\beta(p) = \delta_{\alpha\beta} \,. \tag{19}$$

Inserting Eq. (11) into Eq. (17), we get

$$a_{\beta\alpha} = f(u_\beta^*(p) Q u_\alpha(p)) \,.$$

Combining this with Eq. (16), we have

$$(\Psi_\beta^* U \Psi_\alpha) = f(u_\beta^*(p) Q u_\alpha(p)) \,.$$

Noting Eq. (2) and using general covariance properties, this yields:

$$(\Psi_\beta^* j_\mu(x)\Psi_\alpha) = \frac{1}{V} \cdot f(\bar{u}_\beta(p)\gamma_\mu Q u_\alpha(p)) \,, \tag{20}$$

for states Ψ_α and Ψ_β having the same momentum p. Comparing this with Eq. (13), we note the lack of renormalization and the justification for omitting the meson term F_μ. The same conclusion under the same conditions Eq. (13) and Eq. (14), was obtained also by S. WEINBERG [8], who used a method which was originally given by TAKAHASHI [9] on another problem.

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[8] S. WEINBERG: private communication (1958).
[9] Y. TAKAHASHI: *Nuovo Cimento*, **6**, 371 (1957).

This proof of the lack of renormalization is not interesting in the case of the β-decay problem, since for the vector current, it can be more easily proved ([10]) by analogy with quantum electrodynamics, and for the axial vector part, the possibility of current conservation is ruled out, as already mentioned (*). But a possible application of this theorem is to the strangeness-violating current. For this case, if we neglect the mass-difference between the nucleon and the hyperon, the proof is almost the same as given here.

Finally, it may be worth-while to point out that the condition Eq. (14) is essential for our proof. As is now known ([11]), we can construct a conserved axial vector current if we can neglect the bare pion-mass and if the interaction Hamiltonian for the pion-nucleon system is ps-pv instead of ps-ps. However, in this case, Eq. (14) fails to hold, due to the nature of the ps-pv interaction, and so we cannot prove the lack of renormalization, although we have a conserved current.

4. – Examples of approximately conserved current.

As was explained in Section 2, it is quite unlikely to have a conserved strangeness-violating current. Especially for certain cases, this impossibility was proved in a rather rigorous way. However, as we mentioned there, if we neglect the mass-difference between the nucleon and the hyperon, we may have a conserved current. Here, we give some examples of this approximately conserved strangeness-violating current. Hereafter, we shall assume this neglect of the mass-difference without further mention.

(*) *Note added in proof.* - Actually in case of a conserved axial vector current, $(\Psi_\alpha^* U \Psi_\alpha)$ is rather ambiguous, as we can infer from the work of GOLDBERGER and TREIMAN ([3]). Generally, it contains a term of the form $0 \times \infty$. Thus, in spite of our formal proof, we cannot say anything. For example, we can argue that $(\Psi_\alpha^* U \Psi_\beta)$ is a covariant pseudo-scalar and so it must be identically zero, as was pointed out by Dr. BERNSTEIN. Furthermore, in this connection, it may be worth-while to point out that the conclusion by BLIN-STOYLE (*Nuovo Cimento*, **10**, 132 (1958)) is not correct. He concludes that conservation of an axial vector current does not mean lack of renormalization by computing it in the lowest order perturbation in the case of the Schwinger theory. In it, he uses the real nucleon mass. However, in the Schwinger theory, the bare nucleon mass must be put into zero in order to have conservation. Thus, the real nucleon mass is at least of order of the square of the coupling constant. If we take account of this fact, we find that his formula for the renormalization constant is of much higher order, and so the renormalization constant is unit in the lowest order perturbation in contrast to his claim. The author wishes to thank Drs. WEINBERG and BERNSTEIN for their comments and criticism on these points.

([10]) B. L. IOFFE: *Nuovo Cimento*, **10**, 352 (1958).

([11]) J. C. TAYLOR: quoted in footnote of J. C. POLKINGHORE: *Nuovo Cimento*, **8**, 179 (1958); W. B. TEUTSCH and E. C. G. SUDARSHAN: private communication (1958).

One example is as follows. We assume the global symmetric pion-baryon interaction ([12]) and neglect the K-meson interactions. Then, we can easily show that we have

$$(21) \qquad \begin{cases} \dfrac{\partial}{\partial x_\mu}(\bar{Y}\gamma_\mu N) = 0\,, \\ \dfrac{\partial}{\partial x_\mu}(\bar{Z}\gamma_\mu N) = 0\,, \\ \dfrac{\partial}{\partial x_\mu}(\bar{Y}\gamma_\mu Z) = 0\,, \end{cases}$$

where

$$N = \begin{pmatrix} p \\ n \end{pmatrix}, \qquad Y = \begin{pmatrix} \Sigma_+ \\ Y_0 \end{pmatrix}, \qquad Z = \begin{pmatrix} Z_0 \\ \Sigma_- \end{pmatrix}.$$

Since $(\bar{Y}\gamma_\mu N)$ does not change the charge, this cannot be the interaction responsible for the lepton decay of the K-meson. Thus, the vector part of a possible interaction Hamiltonian for K_{e_3} is (*)

$$(22) \qquad H_1^{(v)} = G(\bar{Z}\gamma_\mu N)(\bar{\nu}\gamma_\mu(1+\gamma_5)e) + \text{H. C.}$$

It is interesting to observe that

$$(23) \qquad j_\mu = (\bar{Z}\gamma_\mu N) = \frac{1}{\sqrt{2}}(\bar{\Lambda}_0 + \bar{\Sigma}_0)\gamma_\mu p + (\bar{\Sigma}_-\gamma_\mu n)\,,$$

behaves like an isotopic spinor in the usual assignment of the isotopic spin for baryons.

Thus, the interaction Eq. (22) satisfies the criterion given previously by various authors ([13]). So, if the K-meson is pseudoscalar and Eq. (16) is responsible for the K_{e_3} and a similar one for the K_{μ_3} modes, then we can calculate absolute transition probabilities for $K_2^0 \to \pi^\pm + e^\mp \pm \nu$ and $K_2^0 \to \pi^\pm + \mu^\mp \pm \nu$ from the known decay of K^+, which are not inconsistent with experiment. Thus, if we forget the approximate conservation law Eq. (21), the interaction Hamiltonian Eq. (22) is quite interesting. Actually, the approximation to get the conserved current in this case is a very bad one for the decay of the K-meson, since we neglected the K-meson interaction.

(*) *Note added in proof.* - We may add $(\bar{\Xi}\gamma_\mu Y)$ to $(\bar{Z}\gamma_\mu N)$ without changing any conclusion given below. $(\bar{\Xi}\gamma_\mu Y)$ is also an iso-spinor.

([12]) M. Gell-Mann: *Phys. Rev.*, **106**, 1296 (1957).

([13]) S. Okubo, R. E. Marshak, E. C. G. Sudarshan, W. B. Teutsch and S. Weinberg: *Phys. Rev.*, **112**, 665 (1958); I. Y. Kobzarev and L. B. Okun: private communication; L. B. Okun: *Proceedings of the Padua-Venice Conference* (1958); S. Okubo, R. E. Marshak and E. C. G. Sudarshan: *Phys. Rev. Lett.*, **2**, 13 (1959).

Another example of conserved vector current is possible in the case of Sakata's model [14]. In this model, only Λ, the neutron and the proton are the fundamental particles; the others like the pion, the kaon and the Σ are supposed to be composite particles. If in this model the interaction Hamiltonian for the strong processes is given by

$$H_1 = \sum_\alpha g_\alpha \{2(\bar{\Lambda} Q_\alpha \Lambda)(\bar{N} Q_\alpha N) + (\bar{\Lambda} Q_\alpha \Lambda)(\bar{\Lambda} Q_\alpha \Lambda) + (\bar{N} Q_\alpha N)(\bar{N} Q_\alpha N)\} \tag{24}$$

then, we have

$$\frac{\partial}{\partial x_\mu}(\bar{\Lambda}\gamma_\mu N) = 0 \,. \tag{25}$$

In Eq. (24) Q_α stands for either 1 $i\gamma_5$, γ_μ, $i\gamma_5\gamma_\mu$ or $(i/4)[\gamma_\mu, \gamma_\nu]$ and in the derivation of Eq. (25) the mass-difference between Λ and N is neglected. The interaction Eq. (24) has an interesting symmetry. In it Λ, the neutron and the proton constitute a triplet instead of a nucleon doublet and a Λ singlet. Thus insofar as we might neglect the mass-difference between Λ and the nucleon we cannot distinguish between Λ and the nucleon. Hence if Eq. (24) is really a true representation of nature the mass-difference between Λ and the nucleon must be quite important and the conservation law Eq. (25) will be very poorly satisfied.

However if we forget how we obtained Eq. (24) and look for the symmetry between Λ and the neutron this presents an interesting feature in view of the fact that the muon and the electron seem to enjoy a similar relationship. Namely the muon and the electron behave almost in the same fashion in all the known interactions except for the mass-difference which we do not really understand. Analogously we may look for the same situation in connection with Λ and the neutron. In the weak interactions this symmetry between Λ and the neutron is satisfied if we assume the usual universal interaction for four fermions. The extension of this analogy to the strong interactions leads us to Sakata's model and Eq. (24). This analogy between the muon-electron pair and the Λ-neutron pair may however be too speculative and certainly requires further investigation [15].

Finally we comment that it seems to be very difficult to derive a conserved axial vector current for the strangeness-violating processes. Thus far we have not succeeded in doing so.

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[14] S. SAKATA: *Prog. Theor. Phys.*, **16**, 686 (1956); L. B. OKUN: *Geneva Conference on High Energy Physics: CERN* (1958) p. 223.

[15] A. similar idea was presented by M. GOLDHABER: *Phys. Rev. Lett.*, **1**, 467 (1958). However, this idea is suitable in the case of the Sakata model, since otherwise we must double the fermions, as M. GOLDHABER proposed.

5. – Discussions.

Thus we have investigated the consequences of the conserved current. As we have found in the previous sections in the case of the strangeness-violating current it seems unlikely to have a conserved current whether it is vector or axial-vector. In the case of the axial vector current in the strangeness-conserving process the idea of the conserved axial vector current is ruled out by experiment as was stated in the beginning. Then a question naturally arises. Why is the vector part of the strangeness-conserving current the only one which is conserved? In the Feynman-Gell-Mann scheme ([2]) we add rather artificially a counter mesonic term in order to get the conserved current. However in the Sakata-Okun ([14]) model the conservation of the strangeness-conserving vector current follows quite naturally without any *ad-hoc* assumption. Furthermore it is possible to make the whole theory chiral-invariant ([13]). whereas in the usual theory the strong interactions are not. However in the case of the pion-nucleon interaction STECH ([16]) succeeded in constructing a chiral-invariant ps-pv theory which is in essence equivalent to the usual ps-pv theory. His method takes advantage of the conservation of the vector current and thus it will be quite difficult to extend his method to the case of the hyperon-kaon interaction, since we may not construct a conserved current in this case. These might be indications that the Sakata-Okun model is more natural than the usual theory.

Finally we like to mention another viewpoint on the conservation of the vector current in the case of the strangeness-conserving process. This view point is to regard the conservation as rather accidental. For example in Sugawara's ([17]) scheme for the weak interactions he postulates the existence of boson pairs together with the baryon pairs to be coupled together. Thus we will have a pion-pion current and a kaon-pion current and the former constitutes the conserved vector current together with the corresponding nucleon current. However his theory predicts some unfavorable things like a large probability for $K_1^0 \to \pi^{\pm} + e^{\mp} \pm \nu$ and gives the change of the strangeness by two units, which would give a big mass-difference ([18]) between K_1^0 and K_2^0 and hence may be excluded by experiments ([19]).

([16]) B. STECH: *Phys. Rev. Lett.*, **2**, 63 (1959).
([17]) M. SUGAWARA: *Gatlinburg Conference, Tennessee* (October 1958).
([18]) L. B. OKUN and B. PONTECORVO: *Žu. Èksper. Teor. Fiz.*, **32**, 1587 (1957).
([19]) E. BOLDT, D. CALDMELL and Y. PAL: *Phys. Rev. Lett.*, **1**, 151 (1958).

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RIASSUNTO (*)

Si esamina la relazione tra la mancanza di rinormalizzazione e la conservazione della corrente. Si prova, inoltre, che in determinate condizioni non possiamo costruire una corrente che si conservi in un processo che violi la stranezza. Tuttavia, trascurando le differenze di massa tra i barioni, lo si può fare, e se ne danno alcuni esempi. Si espongono, infine, alcune considerazioni sulle interazioni deboli.

(*) *Traduzione a cura della Redazione*

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Higher Born Approximations in Non-Relativistic Coulomb Scattering.

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Summary. — In this paper we calculate the first three Born approximations to the differential cross-section for non-relativistic scattering by a Yukawa potential in the limit of zero screening. The result agrees with the exact (Rutherford) cross-section for Coulomb scattering. This supports the suggestion made by Dalitz that the (divergent) higher Born approximations for Coulomb scattering act solely as a phase-factor multiplying the first Born approximation matrix-element.

1. – Introduction.

The exact solution of the Schrödinger equation for scattering in a Coulomb potential was first given by GORDON (1) who found the asymptotic form

$$\psi(\boldsymbol{r}) \sim \left[1 - \frac{\gamma^2}{ik(r-z)}\right] \exp\left[ikz + i\gamma \log k(r-z)\right] - \\ - \frac{\Gamma(1+i\gamma)}{\Gamma(1-i\gamma)} \gamma \operatorname{cosec}^2 \frac{\theta}{2} \cdot \frac{1}{kr} \exp\left[ikr - i\gamma \log k(r-z)\right],$$

where $\gamma = ZZ'e^2/4\pi\hbar v$, $k = mv/\hbar$ and θ is the scattering angle. This leads to the well-known Rutherford differential cross-section.

If an attempt is made to obtain this result by means of the Born approximation method, it is found that the first approximation leads to the exact result, and that the higher approximations are all infinite (as a consequence

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(1) W. GORDON: *Zeits. f. Phys.*, **48**, 180 (1928).

of the infinite forward scattering amplitude in the first Born approximation). DALITZ (2) has suggested that if the Born approximation series for the scattering amplitude is evaluated for a screened (Yukawa) potential, then in the limit of zero screening it will reduce to the first Born approximation amplitude multiplied by an (infinite) phase factor of unit modulus, and so will lead to the exact result for the differential cross-section.

DALITZ calculated the most divergent parts of the first three approximations and showed that they were equivalent to a phase factor, as suggested. In the present paper the first three approximations are calculated exactly, in the zero-screening limit, and again we find that these lead to the exact result, to this order in Z.

2. – Formulation of the problem.

We require to solve the Schrödinger equation

$$\frac{1}{2m}(-\nabla^2 - p^2)\psi(\boldsymbol{r}) = \frac{Ze^2}{4\pi}\frac{\exp[-\lambda r]}{r}\psi(\boldsymbol{r}),$$

for scattering of an electron by a nuclear charge Ze. Note that we take $\hbar = 1$, and work with rationalized units. λ is the screening constant, and we work in the limit $\lambda \to 0$. We impose the boundary condition

$$\psi(\boldsymbol{r}) \sim \exp[i\boldsymbol{p}\cdot\boldsymbol{r}] + \frac{1}{r}g(\theta)\exp[ipr].$$

Note that this asymptotic behaviour disagrees with that in a pure Coulomb field, as given by GORDON. This is probably the root cause of the difficulties encountered in the Born approximation method for a Coulomb potential. However for a screened potential this is the correct behaviour to impose at infinity.

We then get the integral equation

$$\psi(\boldsymbol{r}) = \exp[i\boldsymbol{p}\cdot\boldsymbol{r}] + \frac{1}{4\pi}\int \mathrm{d}^3r' \frac{\exp[ip|\boldsymbol{r}-\boldsymbol{r}'|]}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\frac{2mZe^2}{4\pi}\cdot\frac{\exp[-\lambda r']}{r'}\cdot\psi(\boldsymbol{r}').$$

We transform to momentum space by defining

$$\varphi(\boldsymbol{s}) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3r \exp[-i\boldsymbol{s}\cdot\boldsymbol{r}]\psi(\boldsymbol{r}),$$

(2) R. H. DALITZ: *Proc. Roy. Soc.*, A **206**, 509 (1951).

which leads to the equation

$$\varphi(\boldsymbol{s}) = \delta(\boldsymbol{s}-\boldsymbol{p}) + \frac{2mZe^2}{s^2-p^2-i\mu}\int \mathrm{d}^3q\, \frac{(2\pi)^{-3}}{(\boldsymbol{s}-\boldsymbol{q})^2+\lambda^2}\,\varphi(\boldsymbol{q})\,.$$

Here μ is a positive infinitesimal introduced in order to define uniquely the Fourier transform of $r^{-1}\exp[ipr]$. The limit $\mu \to 0+$ is to be taken before the limit $\lambda \to 0$.

We look for a solution of the form

$$\varphi(\boldsymbol{s}) = \delta(\boldsymbol{s}-\boldsymbol{p}) + \frac{1}{s^2-p^2-i\mu}\, f(\boldsymbol{s})\,,$$

where $f(\boldsymbol{s})$ is a power series in $\gamma = 2mZe^2/(2\pi)^3$, that is

$$f(\boldsymbol{s}) = \sum_{n=1}^{\infty} \gamma^n f_n(\boldsymbol{s})\,.$$

We then obtain the equations

(1) $$f_{n+1}(\boldsymbol{s}) = \int \mathrm{d}^3q\, \frac{1}{(\boldsymbol{s}-\boldsymbol{q})^2+\lambda^2}\cdot\frac{1}{q^2-p^2-i\mu}\cdot f_n(\boldsymbol{q})\,,$$

with

(2) $$f_1(\boldsymbol{s}) = \frac{1}{(\boldsymbol{s}-\boldsymbol{p})^2+\lambda^2}\,.$$

These equations determine the solution completely. For on returning to configuration space we obtain

$$\psi(\boldsymbol{r}) = \exp[i\boldsymbol{p}\cdot\boldsymbol{r}] + \int \mathrm{d}^3s\, \frac{\exp[i\,\boldsymbol{s}\cdot\boldsymbol{r}]}{s^2-p^2-i\mu}\, f(\boldsymbol{s})\,,$$

which has the asymptotic form

$$\psi(\boldsymbol{r}) \sim \exp[i\boldsymbol{p}\cdot\boldsymbol{r}] + 2\pi^2 f(p\hat{\boldsymbol{r}})\,\frac{\exp[ipr]}{r}\,,$$

where $\hat{\boldsymbol{r}}$ is a unit vector along $\boldsymbol{r}$. The differential cross-section for scattering into the direction of $\hat{\boldsymbol{r}}$ is then given by

$$\frac{\mathrm{d}\sigma(\hat{\boldsymbol{r}})}{\mathrm{d}\Omega} = (2\pi^2)^2\,|f(p\hat{\boldsymbol{r}})|^2\,.$$

The exact (Rutherford) cross-section is actually given by

$$|f_R(p\hat{\boldsymbol{r}})| = \frac{\gamma}{p^2(\hat{\boldsymbol{p}} - \hat{\boldsymbol{r}})^2},$$

which is identical with $f_1(p\hat{\boldsymbol{r}})$ in the limit $\lambda \to 0$.

Hence if the Born approximation result is to be valid we require

$$\operatorname*{Lim}_{\lambda\to 0} \gamma^2 f_1^* f_1 = \operatorname*{Lim}_{\lambda\to 0} \left(\sum_{n=1}^{\infty} \gamma^n f_n^*\right)\left(\sum_{m=1}^{\infty} \gamma^m f_m\right)$$

or, equating powers of γ,

$$f_1 f_2^* + f_2 f_1^* = 0, \tag{3}$$

$$f_1 f_3^* + f_2 f_2^* + f_3 f_1^* = 0 \tag{4}$$

and generally

$$f_1 f_n^* + f_2 f_{n-1}^* + \ldots + f_{n-1} f_2^* + f_n f_1^* = 0. \tag{5}$$

MØLLER [3] showed that for $\lambda \to 0$ f_2 is pure imaginary, and behaves like $\log \lambda$. Since f_1 is real, (3) does indeed hold. DALITZ considered the most divergent part of f_3, which behaves like $(\log \lambda)^2$. He showed that the parts of (4) which behave like $(\log \lambda)^2$ actually cancel.

In this paper we calculate f_3 exactly (in the limit $\lambda \to 0$) and show that (4) is satisfied.

In order to clarify the calculation we first evaluate $f_2(\boldsymbol{s})$ in fair detail, and then outline the evaluation of $f_3(\boldsymbol{s})$. The general method can be summarized as follows. We require to evaluate integrals over either one or two 3-momenta (for f_2 and f_3 respectively). By using the Feynman identity

$$\frac{1}{ab} = \int_0^1 \frac{\mathrm{d}x}{[ax + b(1-x)]^2},$$

we can simplify the denominators in the integrands, and so evaluate the momentum integrals (this is done in the Appendix). We are then left with the integrals over the auxiliary variables x, or x and y which we have introduced. In both cases the integrands involve the square root of a function A_x^2 which changes sign at some point in the range of integration.

We therefore split the range of integration at this point. On one side we

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[3] C. MØLLER: *Zeits. f. Phys.*, **66**, 513 (1930).

make a trigonometrical substitution, on the other side a hyperbolic one. We can then perform the integral, or the first of the two integrals in the case of f_3. This introduces a Log which is then identified with a principal value log. This completes the evaluation of f_2. In the case of f_3 a further integration has to be performed which is straightforward.

3. – The calculation.

Throughout the following the limits $\mu \to 0$ and $\lambda \to 0$ are to be understood.

Consider

$$f_2(\boldsymbol{s}) = \int \mathrm{d}^3 q \, \frac{1}{(\boldsymbol{s}-\boldsymbol{q})^2 + \lambda^2} \cdot \frac{1}{q^2 - p^2 - i\mu} \cdot \frac{1}{(\boldsymbol{q}-\boldsymbol{p})^2 + \lambda^2} \,. \tag{6}$$

From (A.3) we get

$$f_2(\boldsymbol{s}) = \pi^2 i \int_0^1 \mathrm{d}x \, \frac{1}{A_x [B^2 - (A_x + i\lambda)^2]} \,, \tag{7}$$

where

$$\boldsymbol{B} = x\boldsymbol{p} - \boldsymbol{s} = \boldsymbol{\rho} - (1-x)\boldsymbol{p} \,; \qquad \boldsymbol{\rho} = \boldsymbol{p} - \boldsymbol{s}$$

and

$$A_x^2 = p^2(1-x)^2 - x\lambda^2 + (1-x)i\mu \,.$$

Now $A_x^2 = 0$ has roots at

$$1 - x = \pm \frac{\lambda}{p} - \frac{i\mu}{2p^2} + O(\lambda^2, \mu^2) \,.$$

Since we are only interested in the range $0 \leqslant x \leqslant 1$ we can replace these roots by

$$1 - x = \pm \left(\frac{\lambda}{p} - i\eta \right) , \qquad \eta \to 0+ \,;$$

that is

$$A_x^2 \doteq p^2 [(1-x)^2 - (\alpha - i\eta)^2] \,; \qquad \alpha = \frac{\lambda}{p} \,.$$

The $i\eta$ can now be dropped, giving

$$\begin{cases} A_x = p\sqrt{(1-x)^2 - \alpha^2} & \text{for } 1 - x \geqslant \alpha, \\ \quad = ip\sqrt{\alpha^2 - (1-x)^2} & \text{for } 1 - x \leqslant \alpha. \end{cases} \tag{8}$$

We can now evaluate (7) by splitting the range of integration at $1-x=\alpha$. In the range $0 \leqslant x \leqslant 1-\alpha$ we substitute $1-x=\alpha\cosh u$; then

$$\int_0^{1-\alpha} \frac{\mathrm{d}x}{A_x} f(x, A_x) = \int_0^{\cosh^{-1}(1/\alpha)} \frac{\mathrm{d}u}{p} f((1-\alpha\cosh u,\ p\alpha\sinh u)\,.$$

Similarly for the range $1-\alpha \leqslant x \leqslant 1$ we substitute $1-x=\alpha\sin u$, giving

$$\int_{1-\alpha}^{1} \frac{\mathrm{d}x}{A_x} f(x, A_x) = \int_0^{\pi/2} \frac{\mathrm{d}u}{ip} f(1-\alpha\sin u,\ ip\alpha\cos u)\,.$$

In the first range the substitution gives

$$B^2-(A_x+i\lambda)^2 = a+b\cosh u+c\sinh u$$

with $a=\varrho^2+2\alpha^2p^2$, $b=-\alpha\varrho^2$ and $c=-2i\alpha^2p^2$. We are interested in the limit $\alpha\to 0$; none-the-less we cannot ignore b and c with respect to a since $\cosh u$ and $\sinh u$ are $O(\alpha^{-1})$ at the upper end of the range. We use the result

$$\int \frac{\mathrm{d}u}{a+b\cosh u+c\sinh u} =$$

$$= \frac{1}{\sqrt{a^2+c^2-b^2}}\left\{\mathrm{Log}\left[(a-b)\,\mathrm{tgh}\frac{u}{2}-c+\sqrt{a^2+c^2-b^2}\right]-\right.$$

$$\left.-\mathrm{Log}\left[-(a-b)\,\mathrm{tgh}\frac{u}{2}+c+\sqrt{a^2+c^2-b^2}\right]\right\}.$$

Working to the lowest non-vanishing order in α we find

$$(a-b)\,\mathrm{tgh}\left(\frac{1}{2}\cosh^{-1}\frac{1}{\alpha}\right)-c+\sqrt{a^2+c^2-b^2} = 2\varrho^2\,,$$

$$-(a-b)\,\mathrm{tgh}\left(\frac{1}{2}\cosh^{-1}\frac{1}{\alpha}\right)+c+\sqrt{a^2+c^2-b^2} = -2i\alpha^2p^2\,,$$

$$-c+\sqrt{a^2+c^2-b^2} = +c+\sqrt{a^2+c^2-b^2} = \varrho^2$$

and

$$\sqrt{a^2+c^2-b^2} = \varrho^2\,.$$

As u goes along the real axis the argument of each Log goes along a *straight* line in the complex plane from its initial to its final value. Hence each Log must *separately* be taken as a principal value log. We therefore get

$$\int_0^{1-\alpha}\mathrm{d}x\,\frac{1}{A_x[B^2-(A_x+i\lambda)^2]} \to \frac{1}{p\varrho^2}\left(\frac{i\pi}{2}+\log\frac{\varrho^2}{\alpha^2p^2}\right).$$

In the second range we can immediately replace α by 0. We then get

$$\int_{1-\alpha}^{1} \mathrm{d}x \frac{1}{A_x[B^2-(A_x+i\lambda)^2]} \to \int_0^{\pi/2} \frac{\mathrm{d}u}{ip\varrho^2} = \frac{\pi}{2ip\varrho^2}\,.$$

Therefore finally

$$f_2(\boldsymbol{s}) = \frac{\pi^2 i}{p\varrho^2} \log \frac{\varrho^2}{\alpha^2 p^2}\,; \qquad \boldsymbol{\varrho} = \boldsymbol{p} - \boldsymbol{s}\,, \qquad \alpha = \frac{\lambda}{p}\,. \tag{9}$$

This result agrees with previous calculations (for references to numerous such calculations cf. DALITZ [2]).

We now turn to $f_3(\boldsymbol{s})$. From (A.4) we have

$$f_3(\boldsymbol{s}) = (\pi^2 i)^2 \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \, \frac{1}{A_x A_y} \cdot \frac{1}{D^2-(A_x+A_y+i\lambda)^2}\,, \tag{10}$$

where

$$\boldsymbol{D} = x\boldsymbol{p} - y\boldsymbol{s} = \boldsymbol{\varrho} - (1-x)\boldsymbol{p} + (1-y)\boldsymbol{s}\,; \qquad \boldsymbol{\varrho} \equiv \boldsymbol{p} - \boldsymbol{s}$$

and

$$A_x^2 = p^2(1-x)^2 - x\lambda^2 + (1-x)i\mu\,,$$

$$A_y^2 = s^2(1-y)^2 - y\lambda^2 + (1-y)i\mu\,.$$

Note that $s^2 = p^2$.

We again make the approximation given by (8), for both A_x and A_y; and we again split the integration ranges. Thus

$$\left\{\begin{aligned} & f_3(\boldsymbol{s}) = \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \ldots \equiv E + F + G \\ & \text{where} \\ & E = \int_0^{1-\alpha} \mathrm{d}x \int_0^{1-\alpha} \mathrm{d}y \ldots \\ & F = 2\int_{1-\alpha}^{1} \mathrm{d}x \int_0^{1-\alpha} \mathrm{d}y \ldots \\ & G = \int_{1-\alpha}^{1} \mathrm{d}x \int_{1-\alpha}^{1} \mathrm{d}y \ldots\,. \end{aligned}\right. \tag{11}$$

Note that the symmetry properties of f_3 with respect to x and y have been used for the F term (see diagram).

Consider E. We make the substitutions $1-x=\alpha\cosh u$, $1-y=\alpha\cosh v$. Then we find

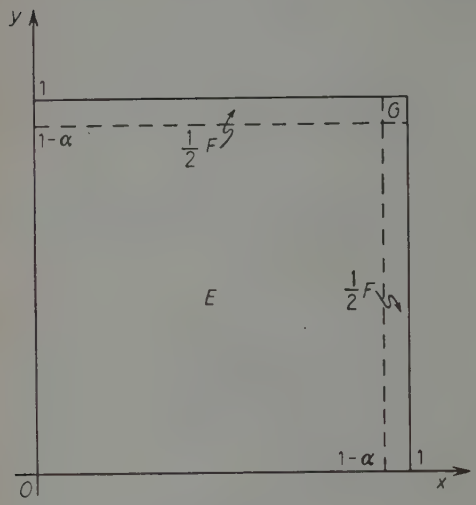

Fig. 1.

$$E=\frac{(\pi^2 i)^2}{p^2}\int_0^{\cosh^{-1}(1/\alpha)}\mathrm{d}u\cdot\int_0^{\cosh^{-1}(1/\alpha)}\frac{\mathrm{d}v}{a+b\cosh v+c\sinh v}, \tag{12}$$

where

$$a=\varrho^2-\varrho^2\alpha\cosh u-2ip^2\alpha^2\cdot\sinh u+3p^2\alpha^2$$

$$b=-\varrho^2\alpha-(2p^2-\varrho^2)\alpha^2\cosh u$$

and

$$c=-2\alpha^2p^2\sinh u-2i\alpha^2p^2.$$

In the v integral we make the substitution $t=\mathrm{tgh}\,(v/2)$ and define $T=\mathrm{tgh}\,(\frac{1}{2}\cosh^{-1}1/\alpha)$.

We find

$$\int_0^{\cosh^{-1}(1/\alpha)}\frac{\mathrm{d}v}{a+b\cosh v+c\sinh v}=\frac{1}{\sqrt{a^2+c^2-b^2}}\Big\{\mathrm{Log}\,[a+b+(c+\sqrt{a^2+c^2-b^2})t]-\mathrm{Log}\,[a+b+(c-\sqrt{a^2+c^2-b^2})t]\Big\}\Big|_0^T.$$

Here again the Logs must be taken as principal values logs. We thus have

$$\int_0^{\cosh^{-1}(1/\alpha)}\frac{\mathrm{d}v}{a+b\cosh v+c\sinh v}=\frac{1}{\sqrt{a^2+c^2-b^2}}\left\{\log\left[\frac{a+b+(c+\sqrt{a^2+c^2-b^2})T}{a+b}\right]-\log\left[\frac{a+b+(c-\sqrt{a^2+c^2-b^2})T}{a+b}\right]\right\}.$$

We *cannot* immediately write this as a single principal value log (for consider the trivial example

$$\log\exp\left[\frac{2}{3}\pi i\right]-\log\exp\left[-\frac{2}{3}\pi i\right]\neq\log\left[\frac{\exp[\frac{2}{3}\pi i]}{\exp[-\frac{2}{3}\pi i]}\right]\Big).$$

However we now show that such a reduction is possible in the present case. By direct substitution, with $T=1-\alpha+(\alpha^2/2)+O(\alpha^4)$, we find that

$$(14)\qquad \sqrt{a^2+c^2-b^2}=\varrho^2\left(1-\frac{\alpha^2}{2}+\frac{2\alpha^2p^2}{\varrho^2}\right)(1-\alpha\cosh u)- \\ -2ip^2\alpha^2(\alpha\sinh u)+p^2\alpha^2+O(\alpha^3)\,,$$

$$\beta\equiv a+b=\varrho^2(1-\alpha)(1-\alpha\cosh u)-2p^2\alpha(\alpha\cosh u+i\alpha\sinh u)+3p^2\alpha^2\,,$$

$$(15)\qquad \gamma\equiv a+b+(c+\sqrt{a^2+c^2-b^2})T= \\ =2(1-\alpha)\varrho^2(1-\alpha\cosh u)-2p^2\alpha(\alpha\cosh u)+ \\ +2(1-2i)p^2\alpha(\alpha\sinh u)+O(\alpha^3)$$

and

$$(16)\qquad \delta\equiv a+b+(c+\sqrt{a^2+c^2-b^2})T= \\ =-2p^2\alpha(\alpha\cosh u+\alpha\sinh u)- \\ -2p^2\alpha^2[i-\alpha\cosh u-(1-i)\alpha\sinh u]+O(\alpha^3)\,.$$

In working to the lowest non-vanishing order in α, it must be remembered that u is an integration variable and takes all values in the range $0\leqslant u\leqslant \cosh^{-1}(1/\alpha)$. Hence $\alpha\cosh u$ and $\alpha\sinh u$ can be of order 1, and in particular $1-\alpha\cosh u$ can vanish. However it is possible to make the approximation

$$(16')\qquad \delta\doteq -2p^2\alpha^2(\cosh u+\sinh u+i)=-2p^2\alpha^2(e^u+i)\,.$$

We now prove that

$$(17)\qquad \log\left(\frac{\gamma}{\beta}\right)-\log\left(\frac{\delta}{\beta}\right)=\log\left(\frac{\gamma}{\delta}\right)$$

for all u in $0\leqslant u\leqslant\cosh^{-1}(1/\alpha)$.

Consider first $u=O(1)$. Then we see that

$$\arg\beta\doteq\arg\gamma\doteq 0\quad\text{and}\quad -\tfrac{3}{4}\pi<\arg\delta<-\pi/2$$

so that (17) holds in this case.

Next consider $u=O(\cosh^{-1}(1/\alpha))$. We define

$$\omega=\frac{1}{\alpha}-\cosh u\quad\text{with}\quad\omega\geqslant 0,\qquad\text{and}\qquad\omega=O(1)\,.$$

Then $\sinh u \doteq \cosh u$ and

$$\beta \doteq (1-\alpha)\varrho^2\alpha\omega - 2p^2\alpha^2\left(\frac{1}{\alpha}-\omega\right)(1+i)\,,$$

$$\gamma \doteq 2\beta \qquad \text{and} \qquad \delta \doteq -2p^2\alpha^2\left(\frac{1}{\alpha}-\omega+i\right).$$

It is possible to find an ω_0 of order 1 such that

$$(1-\alpha)\varrho^2\alpha\omega_0 \gg 2p^2\alpha\,.$$

Then for $\omega \geqslant \omega_0$ we see that $\arg\beta \doteq \arg\gamma \doteq 0$ and $\arg\delta \doteq -\pi/2$. Hence in this range, which connects up *continuously* with the range $u = O(1)$, we see that (17) again holds.

Finally for $\omega_0 \geqslant \omega \geqslant 0$, *i.e.* for $(1/\alpha)-\omega_0 \leqslant \cosh u \leqslant 1/\alpha$ we see that $\arg\beta \doteq \arg\gamma$ goes continuously from 0 to $-\frac{3}{4}\pi$ while $\arg\delta$ remains constant at $\doteq -\pi/2$. Thus again (17) holds.

We have therefore shown that

$$\int_0^{\cosh^{-1}(1/\alpha)} \frac{\mathrm{d}v}{\alpha + b\cosh u + c\sinh v} = \frac{1}{\sqrt{a^2+c^2-b^2}}\log\left(\frac{\gamma}{\delta}\right)$$

with $\sqrt{a^2+c^2-b^2}$, γ and δ as given in (14)–(16′). Substituting into (12) we get

$$E = \frac{(\pi^2 i)^2}{p^2}\int_0^{\cosh^{-1}(1/\alpha)} \frac{\mathrm{d}u}{\sqrt{a^2+c^2-b^2}}\log\left(\frac{\gamma}{\delta}\right). \tag{18}$$

We make the substitution $U = e^{-u}$. Throughout the range of integration it is then possible to discard U in comparison with 1 and U^{-1}. Thus we find

$$\sqrt{a^2+c^2-b^2} \doteq \varrho^2(1-\alpha\cosh u) - 2ip^2\alpha^2\sinh u$$
$$\doteq \varrho^2 - U^{-1}(\tfrac{1}{2}\varrho^2\alpha + ip^2\alpha^2)\,,$$

$$\gamma \doteq \varrho^2[1-\tfrac{1}{2}\ U^{-1} - (1+i)p^2\alpha^2U^{-1}/\varrho^2]$$

and

$$\delta \doteq -p^2\alpha^2(U^{-1}+i)\,.$$

Hence

$$E \doteq \frac{(\pi^2 i)^2}{p^2\varrho^2}\int_{\alpha/2}^{1} \frac{\mathrm{d}U}{U-(\frac{1}{2}\alpha + ip^2\alpha^2/\varrho^2)}\log\left\{\frac{\varrho^2[\,U-\frac{1}{2}\alpha - p^2\alpha^2(1+i)/\varrho^2]}{p^2\alpha^2(-1-iU)}\right\}.$$

We see that the principal value log can now be factorized, leading to

$$E \simeq \frac{(\pi^2 i)^2}{p^2\varrho^2}\int_{\alpha/2}^{1}\frac{\mathrm{d}U}{U-(\alpha/2)-(ip^2\alpha^2/\varrho^2)}\left\{i\pi+\log\frac{\varrho^2}{p^2\alpha^2}-\log(1+iU)+\right.$$
$$\left.+\log\left[U-\frac{\alpha}{2}-\frac{p^2\alpha^2(1+i)}{\varrho^2}\right]\right\}.$$

This is easily evaluated by use of

$$\int\frac{\mathrm{d}U}{U-a}=\log(U-a)$$

and

$$\int\frac{\mathrm{d}U}{U-a}\log(U-b)=\frac{1}{2}[\log(U-a)]^2+\mathfrak{L}_2\left(\frac{b-a}{U-a}\right)=$$
$$=\log(a-b)\log(U-a)-\mathfrak{L}_2\left(\frac{U-a}{b-a}\right).$$

Here $\mathfrak{L}_2(z)$ is Euler's dilogarithmic function

$$\mathfrak{L}_2(z)=-\int_0^z\frac{\log(1-t)\,\mathrm{d}t}{t}=\sum_{n=1}^{\infty}\frac{z^n}{n^2}. \tag{19}$$

Note that the above formulae are only valid if

$$\log(U-a)+\log\left(\frac{U-b}{U-a}\right)=\log(U-b)$$

and

$$\log(a-b)+\log\left(\frac{U-b}{a-b}\right)=\log(U-b)$$

(this is most readily seen by explicit differentiation).

Remembering that $\alpha\to 0$, we get finally

$$E=\frac{(\pi^2 i)^2}{p^2\varrho^2}\left\{-\left(i\pi+\log\frac{\varrho^2}{p^2\alpha^2}\right)\log\left(\frac{-i\alpha^2p^2}{\varrho^2}\right)-\right.$$
$$\left.-\frac{1}{2}\left[\log\left(\frac{-i\alpha^2p^2}{\varrho^2}\right)\right]^2-\mathfrak{L}_2(i)+\mathfrak{L}_2(-i)\right\}. \tag{20}$$

The evaluation of F and G (cf. (11)) is very similar to the above, so that we only give the main steps.

For F, we make the substitutions $1-x=\alpha\sin u$ and $1-y=\alpha\cosh v$.

Then

$$F = 2\,\frac{(\pi^2 i)^2}{ip^2}\int_0^{\pi/2} \mathrm{d}u \int_0^{\cosh^{-1} 1/\alpha} \frac{\mathrm{d}v}{a + b\cosh v + c\sinh v}$$

with

$$a = \varrho^2 - \varrho^2\alpha\sin u + 2\alpha^2 p^2\cos u + 3p^2\alpha^2$$

$$b = -\alpha\varrho^2 - (2p^2 - \varrho^2)\alpha^2\sin u$$

and

$$c = -2i\alpha^2 p^2\cos u - 2i\alpha^2 p^2 .$$

We find, using the same notation as before,

$$\sqrt{a^2 + c^2 - b^2} \simeq 2\varrho^2 , \qquad \beta \simeq \varrho^2 , \qquad \gamma \simeq 2\varrho^2$$

and

$$\delta \simeq -2i\alpha^2 p^2(1 + e^{-iu}) .$$

The result corresponding to (17) follows immediately, and hence

$$F \simeq -2\,\frac{(\pi^2 i)^2}{ip^2\varrho^2}\int_0^{\pi/2} \mathrm{d}u \log\left\{(1 + e^{-iu})\left(\frac{-i\alpha^2 p^2}{\varrho^2}\right)\right\} =$$

$$= -2\,\frac{(\pi^2 i)^2}{ip^2\varrho^2}\left\{\frac{\pi}{2}\log\left(\frac{-i\alpha^2 p^2}{\varrho^2}\right) + i\int_1^{-i} \frac{\mathrm{d}U}{U}\log(1 + U)\right\},$$

where $U = e^{-iu}$. Thus finally

$$F = \frac{(\pi^2 i)^2}{p^2\varrho^2}\left\{i\pi\log\left(\frac{-i\alpha^2 p^2}{\varrho^2}\right) + 2\mathfrak{L}_2(+i) - 2\mathfrak{L}_2(-1)\right\} . \tag{21}$$

G is evaluated by the substitutions

$$1 - x = \alpha\sin u , \qquad 1 - y = \alpha\sin v ,$$

giving

$$G \simeq \frac{(\pi^2 i)^2}{-p^2\varrho^2}\int_0^{\pi/2}\mathrm{d}u\int_0^{\pi/2}\mathrm{d}v = -\frac{(\pi^2 i)^2}{\varrho^2 p^2}\left(\frac{\pi}{2}\right)^2 . \tag{22}$$

Now

$$\mathfrak{L}_2(i) + \mathfrak{L}_2(-i) = \tfrac{1}{2}\mathfrak{L}_2(-1)$$

and

$$\mathfrak{L}_2(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12} .$$

We therefore obtain

$$f_3(\boldsymbol{s}) = E + F + G = (\pi^2 i)^2 \frac{2}{p^2 \varrho^2} \left(\log \frac{\alpha p}{\varrho} \right)^2 . \tag{23}$$

If we substitute for f_1, f_2 and f_3 from (2), (9) and (23) we find, in the limit $\alpha \to 0$, that (4) is satisfied.

4. – Conclusion.

We have demonstrated that in the limit $\lambda \to 0$ the Born approximation result agrees with the exact result for the cross-section to order Z^5.

We emphasize that this calculation is *exact* in the sense that throughout it we have kept *all* terms which do not vanish in the limit $\lambda \to 0$. That is, terms like 1 or $\log \lambda$ have not been discarded even when compared with terms like $(\log \lambda)^2$, though terms like $\lambda \log \lambda$ have been dropped.

* * *

This work forms part of a D. Phil. thesis and was carried out at Oxford under the supervision of Dr. HANDEL DAVIES. It is a pleasure to acknowledge the great debt which I owe him.

APPENDIX

Throughout this Appendix (and therefore throughout the paper) *all* square roots are to be taken as having a non-negative real part. If necessary a cut is to be introduced along the negative half of the real axis.

We require to simplify the expressions

$$f_2(\boldsymbol{s}) = \int \mathrm{d}^3 q \, \frac{1}{(\boldsymbol{s}-\boldsymbol{q})^2 + \lambda^2} \cdot \frac{1}{q^2 - p^2 - i\mu} \cdot \frac{1}{(\boldsymbol{q}-\boldsymbol{p})^2 + \lambda^2} \tag{A.1}$$

and

$$f_3(\boldsymbol{s}) = \int \mathrm{d}^3 r \, \frac{1}{(\boldsymbol{s}-\boldsymbol{r})^2 + \lambda^2} \cdot \frac{1}{r^2 - p^2 - i\mu} \cdot f_2(\boldsymbol{r}) . \tag{A.2}$$

We make use of the Feynman identity

$$\frac{1}{ab} = \int_0^1 \mathrm{d}x \frac{1}{[ax + b(1-x)]^2} .$$

Then

$$\frac{1}{[q^2 - p^2 - i\mu][(\boldsymbol{q}-\boldsymbol{p})^2 + \lambda^2]} = \int_0^1 \mathrm{d}x [(\boldsymbol{q}-\boldsymbol{a})^2 - A_x^2]^{-2} =$$

$$= \int_0^1 \mathrm{d}x \frac{\partial}{\partial A_x^2} [(\boldsymbol{q}-\boldsymbol{a})^2 - A_x^2]^{-1} ,$$

where $\boldsymbol{a} = x\boldsymbol{p}$ and

$$A_x^2 = p^2(1-x)^2 - x\lambda^2 + i\mu(1-x) .$$

Similarly

$$\frac{1}{[(\boldsymbol{q}-\boldsymbol{a})^2 - A_x^2][(\boldsymbol{q}-\boldsymbol{r})^2 + \lambda^2]} = \int_0^1 \mathrm{d}z [(\boldsymbol{q}-\boldsymbol{d})^2 + F]^{-2} ,$$

where $\boldsymbol{d} = z\boldsymbol{a} + (1-z)\boldsymbol{r} = \boldsymbol{r} + z\boldsymbol{B}$; $\boldsymbol{B} \equiv \boldsymbol{a} - \boldsymbol{r}$ and $F = z(1-z)B^2 - zA_x^2 + (1-z)\lambda^2$.

We substitute these results into (A.1) and interchange the order of the integrations, performing the $\boldsymbol{q}$ integration first (*).

We get

$$f_2 = \int_0^1 \mathrm{d}x \frac{\partial}{\partial A_x^2} \int_0^1 \mathrm{d}z \int \mathrm{d}^3q \frac{1}{[(\boldsymbol{q}-\boldsymbol{d})^2 + F]^2} =$$

$$= \int_0^1 \mathrm{d}x \frac{\partial}{\partial A_x^2} \int_0^1 \mathrm{d}z \int \frac{\mathrm{d}^3Q}{(Q^2 + F)^2} = \pi^2 \int_0^1 \mathrm{d}x \frac{\partial}{\partial A_x^2} \int_0^1 \mathrm{d}z \frac{1}{F^{\frac{1}{2}}} .$$

(This result is valid whether F is positive or negative, provided $iF^{\frac{1}{2}}$ contains an imaginary part.)

By means of the substitution

$$Z = 2B^2 z + \lambda^2 + A_x^2 - B^2 + 2iF^{\frac{1}{2}}B ,$$

(*) We continually interchange the order of various operations in this work. This is legitimate provided all the integrals are uniformly convergent. The behaviour of the integrands for large q is always $O(q^{-2})$, and the integrands have no singularities provided the limits $\lambda, \mu \to 0$ are taken only after the operations have been performed. Hence the assumption of uniform convergence is reasonable.

we find

$$\int_0^1 \frac{\mathrm{d}z}{F^{\frac{1}{2}}} = \frac{i}{B}\int \frac{\mathrm{d}Z}{Z} = \frac{i}{B}\,\mathrm{Log}\,Z\Big|_{z=0}^{z=1},$$

where the path of integration in the Z plane is given in the diagram, for the two cases $\mathrm{Re}\,A_x > B$ and $\mathrm{Re}\,A_x < B$

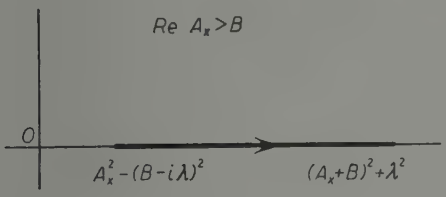

Fig. 2.

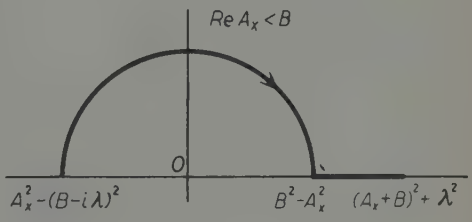

Fig. 3.

Hence

$$\int_0^1 \frac{\mathrm{d}z}{F^{\frac{1}{2}}} = \frac{i}{B}\log\left\{\frac{(A_x+B)^2+\lambda^2}{A_x^2-(B-i\lambda)^2}\right\} = \frac{i}{B}\log\left\{\frac{A_x+B+i\lambda}{A_x-B+i\lambda}\right\},$$

where the *principal value* of the logarithm is to be taken. We therefore obtain

$$f_2(\boldsymbol{s}) = \pi^2 i\int_0^1 \frac{\mathrm{d}x}{A_x[B^2-(A_x+i\lambda)^2]}. \tag{A.3}$$

We now consider f_3. We have

$$\frac{1}{[(\boldsymbol{s}-\boldsymbol{r})^2+\lambda^2][r^2-p^2-i\mu]} = \int_0^1 \mathrm{d}y\,\frac{\partial}{\partial A_y^2}[(\boldsymbol{r}-\boldsymbol{b})^2-A]^{-1}$$

with $\boldsymbol{b} = y\boldsymbol{s}$ and

$$A_y^2 = s^2(1-y)^2 - y\lambda^2 + (1-y)i\mu\,.$$

Hence from (A.2) and (A.3)

$$f_3(\boldsymbol{s}) = \pi^2 i\int \mathrm{d}^3r\int_0^1 \mathrm{d}y\,\frac{\partial}{\partial A_y^2}[(\boldsymbol{r}-\boldsymbol{b})^2-A_y^2]^{-1}\int_0^1 \frac{\mathrm{d}x}{A_x[(\boldsymbol{a}-\boldsymbol{r})^2-(A_x+i\lambda)^2]}.$$

Thus

$$f_3(\boldsymbol{s}) = \pi^2 i \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \, \frac{\partial}{\partial A_y^2} K \,,$$

with

$$K = \int \mathrm{d}^3 r \, [(\boldsymbol{a} - \boldsymbol{r})^2 - (A_x + i\lambda)^2]^{-1} [(\boldsymbol{r} - \boldsymbol{b})^2 - A_y^2]^{-1} \,.$$

We again use the Feynman identity, whence

$$K = \int \mathrm{d}^3 r \int_0^1 \mathrm{d}z \, [(\boldsymbol{r} - \boldsymbol{c})^2 + C]^{-2} \,,$$

where

$$\boldsymbol{c} = z\boldsymbol{a} + (1 - z)\boldsymbol{b} = z\boldsymbol{D} + \boldsymbol{b} \,; \qquad \boldsymbol{D} \equiv \boldsymbol{a} - \boldsymbol{b}$$

and

$$C = z(1 - z) D^2 - z(A_x + i\lambda)^2 - (1 - z) A_y^2 \,.$$

Therefore

$$K = \pi^2 \int_0^1 \mathrm{d}z \, \frac{1}{C^2} = \frac{\pi^2 i}{D} \log \left\{ \frac{A_x + A_y + D + i\lambda}{A_x + A_y - D + i\lambda} \right\} \,,$$

where once again we can show that it is the principal value logarithm which is required. Thus finally

$$f_3(\boldsymbol{s}) = (\pi^2 i)^2 \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \, \frac{1}{A_x A_y [D^2 - (A_x + A_y + i\lambda)^2]} \,. \tag{A.4}$$

RIASSUNTO (*)

Nel presente lavoro si calcolano le tre prime approssimazioni alla sezione d'urto differenziale per lo scattering non relativistico da parte di un potenziale di Yukawa al limite di schermatura zero. Il risultato si accorda con la sezione d'urto esatta (di Rutherford) per lo scattering coulombiano. Ciò avvalora l'ipotesi di Dalitz che le superiori (divergenti) approssimazioni di Born per lo scattering coulombiano agiscano soltanto come fattore di fase che moltiplica l'elemento di matrice della prima approssimazione di Born.

(*) *Traduzione a cura della Redazione.*

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Theoretical Calculations on $\mu^- + {}^{12}C \to {}^{12}B + \nu$.

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(ricevuto il 26 Marzo 1959)

Summary. — Theoretical calculations of the capture rate and possible observations on the recoiling nucleus have been made for the process $\mu^- + {}^{12}C \to {}^{12}B + \nu$. The basic interaction for μ-meson capture is assumed to be the same as that in β-decay. Virtual pion effects which are included with and without the assumption of a conserved vector current, are difficult to observe because of the theoretical uncertainty as to the magnitude of the second forbidden contributions.

1. – Introduction.

A quantitative verification and understanding of the theory of the universal Fermi interaction requires the determination of the coupling constants in the capture process $\mu^- + p \to n + \nu$. Direct methods for this determination [1] using capture in hydrogen are made extremely difficult by the low capture rate and the complications ensuing from the formation of μ-mesic molecules. Recent careful measurements [2,3] of the capture rate $\mu^- + {}^{12}C \to {}^{12}B - \nu$ make it of interest to see to what extent this result may be used.

---

[1] H. Primakoff: *Proc. of the Fifth Annual Rochester Conference on High Energy Physics* (1955), p. 174; K. Huang, C. N. Yang and T. D. Lee: *Phys. Rev.*, **108**, 1340 (1957); I. S. Šapiro, E. I. Dolinsky and L. D. Blohinčev: *Nucl. Phys.*, **4**, 273 (1957); L. Wolfenstein: *Nuovo Cimento*, **7**, 706 (1958).

[2] W. Love, S. Marder, I. Nadelhaft, R. Siegel and A. E. Taylor: *Bull. Am. Phys. Soc.*, **4**, 81 (1959); J. G. Fetkovich, T. H. Fields and R. L. McIlwain: *Bull. Am. Phys. Soc.*, **4**, 81 (1959).

[3] A. D. McGuire, H. V. Argo, F. B. Harrison and H. W. Kruse: *Bull. Am. Phys. Soc.*, **3**, 362 (1958); J. O. Burgman, J. Fischer, B. Leontic, A. Lundby, R. Meunier, J. P. Stroot and J. D. Teja: *Phys. Rev. Lett.*, **1**, 469 (1958).

For this purpose it is necessary to consider carefully the approximations in the original analysis of this experiment given by GODFREY ([4]). We consider also other observables besides the capture rate; in particular, the ^{12}B polarization due to the μ-meson polarization ([5,6]) and possible observations on the ^{12}B recoil direction ([7]). Preliminary results on the capture rate were reported previously ([8]) and detailed calculations have since been given by PRIMAKOFF and FUJII ([9]).

2. – Theory.

The matrix element for μ-meson capture from an initial nuclear state with wave-function $|a\rangle$ to a final state with wave function $|b\rangle$ is

$$(1) \qquad m = \langle b | \sum_i \tau_-(i) \exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\cdot$$

$$\cdot\{V + \boldsymbol{A}\cdot\boldsymbol{\sigma}_i - \boldsymbol{P}\cdot\hat{\boldsymbol{\nu}}\,\boldsymbol{\sigma}_i\cdot\hat{\boldsymbol{\nu}} + \boldsymbol{R}_V\cdot(\boldsymbol{p}_n + \boldsymbol{p})_i + R_A\,\boldsymbol{\sigma}_i\cdot(\boldsymbol{p}_n + \boldsymbol{p})_i\}\varphi_\mu(r_i)\,|a\rangle\,,$$

where

$$V = c_V V_\nu^* V_\mu\,,$$

$$\boldsymbol{A} = (c_A - \mu c_V \nu/2M) V_\nu^* \boldsymbol{\sigma} V_\mu\,,$$

$$\boldsymbol{P} = (c_P - \mu c_V)(\nu/2M) V_\nu^* \boldsymbol{\sigma} V_\mu\,,$$

$$\boldsymbol{R}_V = -(c_V/2M) V_\nu^* \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\nu}}\boldsymbol{\sigma} V_\mu\,,$$

$$R_A = -(c_A/2M) V_\nu^* \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\nu}} V_\mu\,,$$

the subscript i refers to the i-th nucleon, $\boldsymbol{\nu}$ is the neutrino momentum, $\hat{\boldsymbol{\nu}} = (\boldsymbol{\nu}/\nu)$, M is the nucleon mass, φ_μ is the non-relativistic mesic-atom wave function, and V_ν and V_μ are Pauli spinors for the neutrino and μ-meson, respectively. $\boldsymbol{p}$ and $\boldsymbol{p}_n$ are momentum operators acting on the nucleon in initial and final state, respectively; by partial integration, we can write $\boldsymbol{p}_n = \boldsymbol{p} - \boldsymbol{\nu}$, whence our Eq. (1) is seen to be equivalent to (*FP*4). The following assumptions and approximations have been made:

([4]) T. N. K. GODFREY: *Ph. D. Thesis* (Princeton University, 1954).

([5]) J. D. JACKSON, S. B. TREIMAN and H. W. WYLD: *Phys. Rev.*, **107**, 327 (1957).

([6]) W. LOVE, S. MARDER, I. NADELHAFT, R. SIEGEL and A. E. TAYLOR: *Phys. Rev. Lett.*, **2**, 107 (1959).

([7]) T. FULTON: *Nucl. Phys.*, **6**, 319 (1958); S. B. TREIMAN: *Phys. Rev.*, **110**, 44 (1958).

([8]) L. WOLFENSTEIN: *Bull. Am. Phys. Soc.*, **4**, 81 (1959).

([9]) A. FUJII and H. PRIMAKOFF: *Nuovo Cimento*, **12**, 327 (1959). We refer to this reference (indicated as FP) for many details and attempt to omit duplication.

1) The basic interaction is a combination of vector and axial-vector couplings; including virtual pion effects, it is written as Eq. (8) of GOLDBERGER and TREIMAN [10] with the notation $c_A = \sqrt{2}a$, $c_V = \sqrt{2}c$, $C_P = \sqrt{2}m_\mu b$ and $\mu c_V = 2c_V + M\sqrt{2}d$. All the coupling constants are to be evaluated at the correct value of the four-momentum transfer, of course. The C_p term represent the induced pseudoscalar interaction [11] estimated to be about $8C_A$. The μC_V term includes the magnetic-moment type interaction due to virtual pions with a coefficient $\sqrt{2}d$, which has been estimated to be small for the conventional theory [10], but which equals $[(\mu_P - \mu_N)/2M]C_V$ with the assumption of a conserved vector current [12]. The four-component neutrino spinor $u_\nu(1+\gamma_5)/\sqrt{2}$ exists only for left-handed neutrinos, in which case we have replaced it by its top two components V_ν normalized to unity.

2) The μ-meson spinor is treated non-relativistically in that the small components are taken to be zero.

3) Terms in the matrix element involving $(1/M^2)$ have been omitted. The factor $(1/M)$ stands either for (ν/M) or (q/M), which are of the order of 1/10. However, because of the large value of C_P, terms in $|m|^2$ involving $(C_P/M)^2$ and (C_P/M^2) will be included.

4) We consider the nuclear states as collections of physical nucleons each described by a Dirac spinor ignoring possible meson-exchange effects [13].

Using the fact that $|a\rangle$ corresponds to spin zero and $|b\rangle$ to spin one and that there is no change in parity, we select the appropriate irreducible tensor operator in the nuclear matrix element (Eq. (1)) and obtain

$$m = (G_A V_\nu^* \boldsymbol{\sigma} V_\mu - G_P \hat{\boldsymbol{\nu}} V_\nu^* \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\nu}} V_\mu)\cdot\langle b|\boldsymbol{S}|a\rangle\,, \tag{2}$$

$$\langle b|\boldsymbol{S}|a\rangle = \langle b|\sum_i \tau_-(i) j_0(\nu r_i)\varphi_\mu(r_i)\boldsymbol{\sigma}_i|a\rangle\,, \tag{3}$$

$$G_A = C_A(1+x) - (\nu/2M)C_V[\mu(1+x)+\lambda]\,, \tag{4a}$$

$$G_P = (\nu/2M)[C_P(1-2x) - C_A(1+\omega) - C_V\{\mu(1+x)+\lambda\}] + 3xC_A\,, \tag{4b}$$

$$\lambda = \frac{\langle b|\sum_i \tau_-(i)[3j_1(\nu r_i)/\nu r_i]\varphi_\mu(r_i)L_{zi}|a\rangle}{\langle b|S_z|a\rangle}\,,$$

[10] M. L. GOLDBERGER and S. B. TREIMAN: *Phys. Rev.*, **111**, 355 (1958).
[11] L. WOLFENSTEIN: *Nuovo Cimento*, **8** 882 (1958).
[12] R. P. FEYNMAN and M. GELL-MANN: *Phys. Rev.*, **109** 193 (1958); M. GELL-MANN: *Phys. Rev.*, **111**, 362 (1958).
[13] J. S. BELL and R. J. BLIN-STOYLE: *Nucl. Phys.*, **6**, 87 (1958).

$$\omega = \frac{\langle b \,|\, \sum_i \tau_-(i)\,[3j_1(\nu r_i)/\nu r_i)\varphi_\mu(r_i)\,2iz_i\boldsymbol{\sigma}_i\cdot\boldsymbol{p}_i \,|\, a\rangle}{\langle b \,|\, S_z \,|\, a\rangle},$$

$$x = \frac{\langle b \,|\, \sum_i \tau_-(i)\, j_2(\nu r_i)\varphi_\mu(r_i)\,[3\boldsymbol{r}_i\cdot\boldsymbol{\sigma}_i z_i - r_i^2\sigma_{iz}]/2r_i^2 \,|\, a\rangle}{\langle b \,|\, S_z \,|\, a\rangle},$$

where $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}$ is the single-particle angular momentum operator. These « effective couplings » G_A and G_P are the same as those defined in (*FP* 5*c*) (where $\mu = \mu_p - \mu_n + 1$) but with correction terms representing second-forbidden contributions: 1) λ and ω, which arise from $\boldsymbol{R}_V$ and $\boldsymbol{R}_A$, respectively, represent the relativistic terms neglected in FP; 2) x represents the contribution of « d-wave neutrinos ». As is seen from its definition, x corresponds to the $j_2(\nu r_i)$ term in the expansion of the neutrino plane wave; however, it is meaningful to designate this as the « d-wave neutrino » term only when we consider the non-relativistic part of the C_A interaction. In particular, the entire G_P term in Eq. (2) clearly involves d-wave neutrinos also. It may be noted that all terms in Eqs. (4*a*) and (4*b*) apart from $G_A = C_A$ are commonly termed second-forbidden: λ, ω, and x are distinguished from the others in that they represent portions of the nuclear matrix element not reducible to $\langle b \,|\, \boldsymbol{S} \,|\, a\rangle$.

Using Eq. (2) we find the following results:

1) The capture rate for $\mu^- + {}^{12}\mathrm{C} \to {}^{12}\mathrm{B} + \nu$ is given by

$$\tau^{-1} = I_0\,|\langle b \,|\, S_z \,|\, a\rangle|^2\,\nu^2(1+\nu/12M)^{-1}/2\pi\,, \tag{5a}$$

$$I_0 = 3\,|G_A|^2 - 2\,\mathrm{Re}\,G_A^* G_P + |G_P|^2\,. \tag{5b}$$

2) The ^{12}B polarization $\langle \boldsymbol{J} \rangle$ (expectation value of the angular momentum divided by $\hbar$) averaged over recoil directions is given by $\langle \boldsymbol{J} \rangle = J_\mu \boldsymbol{P}_\mu$, where $\boldsymbol{P}_\mu$ is the muon polarization and

$$I_0 J_\mu = 2\,|G_A|^2 - \tfrac{4}{3}\,\mathrm{Re}\,G_A^* G_P\,. \tag{6}$$

3) The ^{12}B recoil angular distribution is proportional to $(1 - AP_\mu \cos\theta)$ where θ is the angle between $\boldsymbol{P}_\mu$ and the recoil and

$$I_0 A = |G_A|^2 + 2\,\mathrm{Re}\,G_A^* G_P - |G_P|^2\,. \tag{7}$$

4) For an unpolarized muon beam, if the recoil is along the z-direction, the relative population of the ^{12}B states are $2\,|G_A|^2$ for $m = -1$, $|G_A - G_P|^2$ for $m = 0$, and zero for $m = 1$. In this case $\langle \boldsymbol{J} \rangle = J_0 \hat{\boldsymbol{\nu}}$ with

$$I_0 J_0 = 2\,|G_A|^2\,. \tag{8}$$

3. – Numerical results and discussion.

The experimental results depend upon the « effective couplings » G_A and G_P and the value of the matrix element $|\langle b|S_z|a\rangle|$. Actually the most one can hope to learn from all these experiments are the values of the two parameters (*) $|G_A\langle b|S_z|a\rangle|$ and (G_P/G_A). We hope to find out whether the results are consistent with the assumption of the universal Fermi interaction and whether they can distinguish between different assumptions about the virtual pion effects. For this purpose we consider four cases: *A*) No virtual pion effects: $C_P = 0$, $\mu = 1$; *B*) The expected virtual pion effects ([10,11]): $C_P = 8\,C_A$, $\mu = 1$; *C*) The virtual pion effects with the added assumption of a conserved vector current ([12]): $C_P = 8\,C_A$, $\mu = 1 + \mu_P - \mu_N$; for these first three cases we assume that C_V and C_A in μ-capture have the same renormalized values as in β-decay in spite of the difference in momentum transfer; in particular, $C_V = .83\,C_A$. We consider in addition a fourth case *D*) which is the same as *B*) except that C_A is replaced by $1.12\,C_A$ so as to give the same capture rate as case *C*).

The theoretical results depend upon estimates of the second-forbidden contributions which are represented by λ, ω and x. For the simplest j-j coupling model in which the ground state of ^{12}C is $(p_{\frac{3}{2}})^4$; $(p_{\frac{3}{2}})^4$, and the ^{12}B state is $(p_{\frac{3}{2}})^3$; $(p_{\frac{3}{2}})^4 p_{\frac{1}{2}}$ and perfect overlap of all the radial p-state functions is assumed, one finds $\lambda = -0.5$, $\omega = -2.7$, and $x = .03$. Since λ and ω enter Eqs. (4) multiplied by $(\nu/2M)$, which is about .05, these three contributions each represent a correction to G_A or G_P of the order of .05 or .10 C_A. (In the evaluation of the capture rate, the « d-wave » term x does not interfere with the main C_A term, but it does give a contribution to I_0 equal to $8(\nu/2M) \times \mathrm{Re}\, C_P^* C_A$.) From a brief examination of other configurations we take as reasonable limits for these quantities: $\lambda = 0 \pm 1.5$, $\omega = -2 \pm 2.5$, and $x = 0 \pm .07$. Using these values (and compounding the theoretical errors statistically) we find the « effective coupling constants » G_A and G_P given in Table I for the four assumptions as to virtual pion effects. The ^{12}B asymmetry parameter A (Eq. (7)), and the ^{12}B polarization parameter J_μ and J_0 (Eqs. (6) and 8) depend only on G_P/G_A and are also listed. The capture rate τ^{-1} is determined by the product of I_0 and the square of the nuclear matrix element $|\langle b|S_z|a\rangle|^2$. As described in detail in FP, the latter is calculated from the nuclear matrix element for the β-decay of ^{12}B (determined from its lifetime) multiplied by the average value of $j_0(\nu r_i)\varphi_\mu(r_i)$. For this average we find $(.75 \pm .06)\ \varphi_\mu(0)$, where the rather arbitrary theoretical uncertainty is primarily due to the uncertainty

(*) We shall assume (G_P/G_A) is real from time-reversal invariance; otherwise there are three real numbers to be determined.

in the accuracy of replacing the integrand in the radial overlap integral for the transition by the stationary charge density distribution and partly due to the uncertainty in the charge distribution of ^{12}C. The resulting capture rate (*) is also shown in Table I.

TABLE I.

Assumption	G_A/C_A	G_P/C_A	$I_0/3C_A^2$	J_μ	A	J_0	τ^{-1}
A	$1.04\pm.09$	$.09\pm.25$	$1.02\pm.12$	$.66\pm.01$	$.42\pm.21$	$.71\pm.11$	$(7.3\pm1.5)\cdot10^3$
B	$1.04\pm.09$	$.48\pm.21$	$.82\pm.12$	$.60\pm.05$	$.75\pm.13$	$.87\pm.07$	$(5.9\pm1.2)\cdot10^3$
C	$1.19\pm.10$	$.63\pm.22$	$1.04\pm.15$	$.58\pm.05$	$.80\pm.10$	$.90\pm.05$	$(7.4\pm1.6)\cdot10^3$
D	$1.16\pm.10$	$.48\pm.23$	$1.04\pm.15$	$.61\pm.04$	$.71\pm.13$	$.86\pm.06$	$(7.4\pm1.6)\cdot10^3$

It is seen from Table I that the theoretical uncertainties make it difficult to detect the virtual pion effects, in particular, to distinguish the consequences of a conserved vector current (that is, distinguish cases B) and C)). An observed capture rate between $6\cdot10^3$ and $7\cdot10^3\ s^{-1}$ would fit all the cases considered. The present data ([2,3]) on the capture rate are hard to interpret because of the differences between different experimental groups and the uncertainty in the amount of capture going to bound excited states of ^{12}C. If it should turn out that the capture rate is greater than $8\cdot10^3\ s^{-1}$, as indicated by the experiments in Ref. ([3]), there would be good evidence either for the « conserved vector current » (case C)) *or* an increased effective axial-vector coupling constant (case D)) *or* the absence of the induced pseudoscalar coupling (case A)). Combinations of these extremes, of course, would also be possible.

The only other experiment on this process so far has been a rough measurement ([6]) of J_μ, the ^{12}B polarization caused by the initial muon polarization. As is seen in Table I, this result is very insensitive to the assumptions, thus to make this experiment would be an excellent test of the muon spin direction and eventually of the ^{12}B depolarization, but not of the process of muon capture. A measurement of the recoil asymmetry parameter A would be of great interest as evidence for parity violation in muon capture and also would show the presence of the induced pseudoscalar interaction if quite accurate results were available (**). The longitudinal polarization J_0 of the recoils from unpolarized

(*) Our mean value for case C is about 5% lower than that of FP, which is mainly due to the choice of a mean $\omega = -2$.

(**) A small d-wave term, while contributing little to the capture rate, may still have a large effect on the asymmetry since it contributes strongly to G_P (Eq. 4*b*). The same is true of the relativistic term ω. Neglecting these terms for case (A) (no virtual pion effects) we get the asymmetry $A = \frac{1}{3}$, which is very different from the values around $\frac{3}{4}$ obtained for the other cases. Small contributions of x and ω, unfortunately, can take away most of this difference.

muons would also show parity violation, and a very large value (around 0.9) accurately measured would also be evidence for the induced pseudoscalar interaction. Even these difficult experiments, however, would not distinguish the « conserved vector current » case *C*) from alternatives *B*) and *D*),

It should be emphasized that the theoretical uncertainties given in this paper are personal estimates of the uncertainties in calculating nuclear matrix elements. A more detailed theoretical examination preferably together with experimental evidence on second-forbidden contributions ([14]) to the β-decay of ^{12}B might serve to decrease these uncertainties.

([14]) J. BERNSTEIN and R. LEWIS: *Phys. Rev.*, **112**, 232 (1958).

Note added in proof.

K. FORD, C. LEVINSON and C. BOUCHIAT have pointed out that the use of a point-nucleus mesic atom wave function for $\varphi_\mu(r)$ is not justified. Use of their wave function obtained by numerical integration reduces the calculated capture rates in Table I by 6%. I wish to thank them for this pre-publication communication.

RIASSUNTO (*)

Per il processo $\mu^- + {}^{12}C \to {}^{12}B + \nu$ sono stati eseguiti calcoli teorici sul tasso di cattura e sulle possibili osservazioni sul rinculo del nucleo. L'interazione fondamentale per la cattura del mesone μ si presume la stessa che nel decadimento β. Effetti pionici virtuali coesistenti con e senza l'assunzione della conservazione di una corrente vettoriale sono difficili da osservare per l'incertezza teorica sulla grandezza dei contributi secondi proibiti.

(*) *Traduzione a cura della Redazione.*

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Time Reversal and Complex Numbers in Quantum Theory.

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Summary. — Time reversal is in a particular position with respect to quantum theory, because of the intervention of antilinear operators. A discussion of this point should explain the role played by complex numbers in quantum theory. It is suggested to look upon that as an effect of an operational irreversibility of time, this being a consequence of the interaction between quantum system and measuring apparatus, when the irreversibility inherent to this is taken into account.

Introduction.

The quantum mechanical theory of time reversal has been dealt with in a wide number of ways and with many different aims, since the earliest paper by Wigner (1), up to recent work on TCP theorem (2-4). Whatever the approach followed, a typical feature was always present: in quantum theory of time reversal the need arises for an operation never occurring in other connections. According to the opinions of the various authors, the operation may be complex conjugation, transposition, or any equivalent device; in all cases we have to deal with a transformation not belonging to the class of unitary transformations. In order to emphasize this circumstance, the word « antiunitary » has been coined.

Though it may appear quite obvious, it is worth-while to note that the theory of TR owes its particularities to the intervention of complex numbers

(1) E. P. Wigner: *Gött. Nachr.* (*Math. Naturw. Klasse*), **31**, 546 (1932).
(2) J. S. Bell: *Proc. Roy. Soc.*, **A 231**, 79 (1955).
(3) W. Pauli: in *Niels Bohr and the Development of Physics* (London, 1955), p. 30.
(4) G. Lüders: *Ann. of Phys.*, **2**, 1 (1957).

in quantum mechanics. This fact suggests the question whether it could be something more than a casual coincidence. Or better: the reason for using complex numbers in quantum theory could be found in our position about time direction. It is the aim of this paper to give an answer to this question, by showing that the need for complex numbers in quantum theory arises from our being unable to reverse time, while wanting to take into account the existence of two time directions.

No attempt will be made in this work of carrying out a relativistic treatment of the subject; this is because new difficulties arise when fields come into consideration, as is necessary in a relativistic theory. Except for Section **5**, where a short discussion of TR for fermions is presented, only systems with a classical analogue are considered. In Section **1** a brief account is given of the arguments from which one can infer the need for a complex quantum theory, while in Section **2** is shown an alternative formulation to the usual one, hereafter called the « real » theory. In Section **3** the matter of antilinear operators is discussed, showing some advantages of the real theory in dealing with them. Section **4** gives the main definitions and results about TR, while Section **5** deals with an extension of these results to fermions. In Section **6**, finally, conclusions are drawn concerning the main theme of this paper, and possible physical consequences are outlined.

1. – Need for a complex theory.

It is a well established fact that the mathematical background for the treatment of quantum theory is given by complex vector spaces. Apart from any theoretical argument, the most convincing proof for the need of complex spaces is to be found in the physical interpretation of the theory: experimental evidence is quite well in favour of the complex theory, as compared with a possible « real » theory. Since the aim of this paper is to point out a connection between the « complex » character of quantum theory and the particular position held in it by time reversal, it will be useful to give a brief account of the experimental evidence for a « complex » against a « real » theory.

All possible proofs for the need of a complex space (see *e.g.* [5] and [6]) eventually lead to the following sentence: a double infinity of states results by superposition from two given distinct states; *i.e.* a particular superposition is precisely specified only by giving *two* real parameters (*e.g.* relative weight and phase of the states). That this is the actual state of affairs is shown by

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([5]) P. A. M. Dirac: *The Principles of Quantum Mechanics*, 3rd ed. (Oxford, 1949).

([6]) A. S. Wightman, V. Bargmann and E. P. Wigner: manuscript circulated at the Varenna Summer School of Physics (1958).

interference of de Broglie waves, by elliptic polarization states of photons (all deriving by superposition from two independent states), by the completeness of the set of eigenstates for the eigenvalues $+\frac{1}{2}$, $-\frac{1}{2}$ of a spin component of a Pauli electron (the double infinity of states wherein a precise value can be given to any other component obtains from these two by superposition). A different argument in favour of the complex theory is given by the essential role which phase factors play in the quantum theory of measurement ([7,8]).

One could also think that some wider number field might be of use in quantum theory: could perhaps quaternions be of help for us to explain some property of elementary particles? The answer may be given by observing that the complex field is the widest commutative field, while commutativity is essential for a vector superposition to be defined. Otherwise, a superposition of superpositions would not be order independent, as it is usually assumed. An extension of the number field would thereby require a deep reinterpretation and a broad reformulation of the theory.

The vector space of quantum theory will be therefore assumed to be a complex space; it is still possible, however, to work out the theory so that —at least formally—only real numbers occur, and thus a real vector space is used. One could think that such a reformulation may have no interesting consequences, on account of its formal character; but it will be shown later that this is not quite true. In order to achieve a better understanding of the question, a sketch of the « real » formulation of quantum theory will be of help.

2. – An alternative formulation: the real theory.

The starting point of the real theory is the following: we can build a real representation of complex numbers through matrices, if we associate to the c.n. $a+ib$ the matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

It is easily shown that the formal properties of c.n.'s are preserved under this mapping, ordinary product becoming matrix product: in fact

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -ad-bc \\ ad+bc & +ac-bd \end{pmatrix}$$

represents

$$(a+ib)(c+id) = (ac-bd)+i(ad+bc).$$

([7]) W. Heisenberg: *I principi fisici della teoria dei quanti*, 2ª ed. (Torino, 1953).
([8]) D. Bohm: *Quantum Theory* (New York, 1951).

If the real representation is applied to operators and state vectors, the following rules will result:

a) the real representative of an operator is a matrix with twice the number of rows and columns than the complex representative of the same operator; an hermitean operator is represented by a symmetrical matrix (note that conjugation of a c.n. means transposition of the corresponding matrix);

b) ket vectors are represented by two-column matrices, bra vectors by two-row matrices, and conjugation of a vector means transposition of its matrix;

c) every product operation between vectors and/or operators becomes a matrix product between representatives; in particular a scalar product of a bra with a ket is represented by a 2×2 matrix (a complex number).

At this point we are naturally led to introduce a real vector space by the following way: let us assume a base in complex vector space S; be it $|1\rangle$, $|2\rangle$, ..., $|n\rangle$, ... We can define a real vector space S_1 having the same base of S: of course S_1 is a subspace of S. Then consider another real space S_2, with two base vectors, $|+\rangle$, $|-\rangle$; S_2 is thus isomorphic to the complex plane, a c.n. corresponding to an operator of S_2, and its representative matrix becoming the representative of this operator in the given base. (In particular, the imaginary unit will give an operator which in the following will be denoted by γ; the characteristic properties of γ are:

(1) $$\gamma|+\rangle = |-\rangle,$$

(2) $$\gamma|-\rangle = -|+\rangle,$$

wherefrom

(3) $$\gamma^2 = -1$$

follows. The c.n. $a+ib$ gives the operator $a+b\gamma$.) Finally, we define the real space S', the topological product of S_1 and S_2; the set $|1+\rangle = |1\rangle|+\rangle$, $|1-\rangle = |1\rangle|-\rangle$, etc., is a base of S'.

With respect to S' the operators of S are still operators, though not of the most general kind, as will be seen later; on the contrary, vectors of S give rise in S' to mixed entities: vectors with respect to S_1, operators to S_2. The scalar product of two vectors of S appears as a scalar product for its effect on S_1, an operator product for its effect on S_2: the result is thus an operator on S_2, a number of S_1 (though, it is always an operator of S'). One could expect to have some trouble therefrom, since one would like to have a scalar product in S transform into a true scalar product in S' as well. However it will now be shown that this is actually true in all cases which are of interest for the theory.

Let $|x\rangle$ be a vector of S, *i.e.* a mixed entity of S'; put

$$|x+\rangle = |x\rangle|+\rangle; \tag{4}$$

this is a vector of S' in a one to one correspondence to $|x\rangle$. For if we put (in S)

$$|x\rangle = \sum_j c_j |j\rangle , \tag{5}$$

where $c_j = a_j + ib_j$ are c.n.'s, it will follow:

$$|x+\rangle = |x\rangle|+\rangle = \sum_j (a_j + b_j\gamma)|j\rangle|+\rangle = \sum_j a_j|j+\rangle + \sum_j b_j|j-\rangle . \tag{6}$$

If $|x\rangle$ is given, then $|x+\rangle$ is uniquely determined; inversely, given $|x+\rangle$, if it is expanded with respect to the base $|j+\rangle$, $|j-\rangle$, the a_j, b_j and therefore $|x\rangle$, are determined. Instead of the mixed entity previously defined, we can therefore associate to every vector of S a true vector of S' (9). The same argument shows then that to one kind of scalar products—the components $\langle j|x\rangle$ of a vector in a given base—scalar products do correspond in S' too —*i.e.* the components $\langle j|x+\rangle$, $\langle j-|x+\rangle$—and vice versa.

The second kind of scalar products of use in the complex theory are those the modulus square of which expresses a transition probability; *i.e.* transition amplitudes. But the squared modulus of a c.n. gives:

$$(a - b\gamma)(a + b\gamma) = a^2 + b^2 ,$$

which is a real number in S' too; thus no difficulty arises from this case.

Apart from some minor and uninteresting detail we saw how to reduce the usual complex theory to a « real » one. We could content ourselves with the real theory for the only fact that in it no entity without a direct physical interpretation, such as complex numbers, occurs; some doubt about the expediency of the new formulation, however, is still justified by some features of it, which shall be briefly considered now.

First, we must note that not every operator of S' does correspond to an observable, and not every hermitean operator either; as far as S_2 is concerned, only the two operators 1 and γ (or any linear combination of them) occur in the theory, while other mathematically possible operators seem to be physically meaningless. Secondly, the phase arbitrarity of the vectors of S causes an analogous arbitrarity in S' (actually in S_2 only). Such arbitrarity amounts

(9) Some arbitrarity is left in the correspondence, because of our arbitrary choice of $|+\rangle$ in (4). We shall see later that a strict connection exists between this fact and the arbitrarity of phase.

to this: *that two vectors differing for a rotation in S_2 are indistinguishable*, or better *that the whole space S_2 is defined to within an arbitrary rotation.* This is because a phase factor $\exp[i\alpha]$ in S produces in S_2 a rotation whose equations are:

$$(7) \qquad \begin{cases} |+\rangle \to \cos\alpha|+\rangle + \sin\alpha|-\rangle\,, \\ |-\rangle \to \cos\alpha|-\rangle - \sin\alpha|+\rangle\,. \end{cases}$$

It follows therefrom that space S_2 is unobservable, as any unitary vector of it may be brought into any other through a rotation. But it appears rather objectionable to introduce a wider space, and then to restrict it with an unobservability postulate; it would be better to keep the old complex space, wherein the phase arbitrarity appears to be much more natural.

All what we have seen hitherto would be, however, of little use if the real theory would show no positive advantage: the aim of this paper, indeed, is to show that the real theory can explain the origin of phase arbitrarity. This will also make clear the reason for using c.n.'s in the usual theory. It will be shown in the following that both facts come from our being tied to one time direction; as an occasional consequence of our discussion it will result a considerable semplification in dealing with time reversal in quantum theory.

3. – Antilinear operators.

It is well known that the quantum theory of time reversal exhibits some difficulties in addition to classical theory, which is already complicated by its lacking of an intuitive meaning ([10]). The reason for the additional « quantum » difficulties seems to be found in the particular role played in TR by complex numbers. The situation was described elsewhere ([11]) by saying that TR is not a true operator; or else ([4]) by defining *ad hoc* a new kind of operators, the *antilinear* ones, the most important (and till now unique) representative of which is TR. One could think that the latter formulation—being more accurate—gives a deeper understanding of the question; it must be noted, however, that by introducing antilinear operators we cause so deep an alteration in the formalism of quantum theory, that we are practically led again to conclude that « an antilinear operator is not a true operator ». Linear and

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([10]) The whole question is extensively and deeply discussed in H. Reichenbach's last book (*The Direction of Time*, Berkeley, 1956) to which one should refer for the proof of some assertions that will be found in the following.

([11]) G. Morpurgo, L. A. Radicati and B. F. Touschek: *Nuovo Cimento*, **12**, 677 (1954).

antilinear operators, indeed, have very few common properties: now I will show, *e.g.*, that if A is antilinear from

$$A^2 = 1$$

one cannot conclude that its eigenvalues are $+1$ and -1, A being thus hermitean. For, if $|a\rangle$ is an eigenvector of A, a the corresponding eigenvalue, we have:

$$|a\rangle = A^2|a\rangle = A \cdot a|a\rangle = \overline{a} \cdot A|a\rangle = \overline{a}a|a\rangle ,$$

i.e.

$$a = \exp[i\alpha] ,$$

with α an arbitrary real number.

If it were not known in advance that only TR gives rise to an antilinear operator, while any other practically interesting operator is linear, it would be impossible for us to keep the present form of quantum theory. LÜDERS gives a clear acknowledgement of this fact when he observes (4) that Dirac's notations are not suited for dealing with antilinear operators. As far as I can see, the main difficulty arises when one tries to define the meaning of $\langle x|A$ where A is an antilinear operator, originarily meaningful only in expressions like $A|y\rangle$.

The usual line of reasoning is as follows: $A|y\rangle$ is a ket, an antilinear function of $|y\rangle$; the scalar product $\langle x| \cdot A|y\rangle$ is then an antilinear function of $|y\rangle$ as well, and thus it cannot be identified with a product $\langle t|y\rangle$, with $\langle t| = \langle x|A$, as it is the case for a linear operator. The same argument may be presented in a rather different way: we can, of course, write:

$$c\langle x| = \langle x|c , \qquad c|y\rangle = |y\rangle c ,$$

with c any complex number. Then, we have obviously:

$$c\langle x| \cdot A|y\rangle = \langle x|c \cdot A|y\rangle = \langle x| \cdot cA|y\rangle = \langle x| \cdot A\overline{c}|y\rangle = \langle x| \cdot A|y\rangle \overline{c} .$$

One may not therefore say that $\langle x| \cdot A|y\rangle$ depends either linearly or antilinearly on $|y\rangle$, as c does not commute with $\langle x| \cdot A|y\rangle$. We must conclude that this expression cannot represent an ordinary c.n.

It appears now a very natural idea that antilinear operators are in some way connected with those « unphysical » operators we met in last section; in fact all this matter will be developed in the following. I would like, however, to anticipate that—at least in my opinion—a result of this study will be that no practical advantage can be found in using antilinear operators: whenever

necessary the real formulation gives the same results, without renouncing to the clearness and simplicity of the usual theory, shown in their highest degree in Dirac's « bra and ket » formalism.

It could be questioned that all this is a matter of notations, and as such a matter of taste; but I would reply that notations may not be quite arbitrary, if they are to serve as a tool for discovering relations and expressing properties of the physical world. I hope that in the following one may find the proof that the real theory, in the sense explained before, is better than the complex theory extended to antilinear operators, though both formulations are exactly equivalent. The real theory, indeed, will permit an easy discussion of the main theme of this paper: the relation between time reversal and complex numbers in quantum theory, while the complex theory hides this relation because of its considering complex numbers like irreducible entities.

4. – Time reversal in the real theory.

Our main concern will be a quantum system with a classical analogue: the extension to systems with no classical analogue will be discussed later.

In defining TR we can adopt two different criteria:

i) to define such a transformation on system states, that transformed states satisfy a Schrödinger equation with the time reversed (this is the way followed by MRT);

ii) to define such a transformation on operators, that the configuration variables $q_1 \dots q_n$ are left unaltered, while the conjugated momenta $p_1 \dots p_n$ are inverted (apart for a term, of no interest for us, of the type $\partial F/\partial q_r$).

As appears from MRT's discussion, both criteria coincide for time-symmetric ([12]) systems; nay, MRT assume ii) as the criterion for time symmetry. On the contrary, I have chosen i) as the definition of TR, so that i) becomes the time symmetry condition; I must now justify this change.

Definition i) is likely to adhere more strictly to the intuitive feeling of TR, as it leads to an equation of motion wherein time « runs backwards »; but the relation between a state and its transform remains altogether undefined, so that no specification for the construction of the transformed state is given, *that be independent from the explicit determination of the transformation operator.* Definition ii), on the contrary, exactly defines the transformed state as that

([12]) I have preferred to introduce a new term, « time-symmetry » instead of the usual one « reversibility », in order to point out that here a symmetry property of the system is involved; the word « reversibility » will be reserved to denote the actual possibility of effecting TR as a physical operation. The distinction is needed, since time symmetry and reversibility have nothing to do with each other.

—to say it easily—in which the system keeps its configuration and gets all its speeds reversed. This enables us to interpret TR as a kind of reference change, *the associated transformation being here defined quite independently from the determination of the transformation operator.* Moreover, as will be shown, our definition is formally simpler.

When so defined, TR is a unitary transformation (antiunitarity is excluded, as we are dealing with a real space) whose operator we will call χ. It must be ([13])

$$\chi^2 = 1 . \tag{8}$$

and then χ is hermitean, with eigenvalues $+1$ and -1.

Put

$$q_r' = \chi q_r \chi^{-1} , \qquad p_r' = \chi p_r \chi^{-1} ; \tag{9}$$

with our definition of TR, it must be:

$$q_r' = q_r , \qquad p_r' = - p_r , \tag{10}$$

i.e. χ commutes with all q's and anticommutes with all p's. This shows that the q's do not form a complete set in Dirac's sense; otherwise χ should be a function of the q's, and one would get:

$$\chi p_r + p_r \chi = 2 p_r \chi + (\chi p_r - p_r \chi) = 2 p_r \chi + \gamma \frac{\partial \chi}{\partial q_r} ;$$

but the last expression does not vanish. Operator χ defines therefore a new degree of freedom, the physical meaning of which will be discussed later.

From the commutation rules

$$q_r p_s - p_s q_r = \delta_{rs} \gamma \tag{11}$$

we have, after a χ-transformation:

$$q_r' p_s' - p_s' q_r' = \delta_{rs} \chi \gamma \chi^{-1} , \tag{12}$$

and using (10):

$$q_r p_s - p_s q_r = - \delta_{rs} \chi \gamma \chi^{-1} . \tag{13}$$

([13]) The validity of relation (8) would deserve a deeper discussion, going beyond the aim of this paper; see however Sect. **5**.

Comparing (13) with (11) we find:

$$\gamma = -\chi\gamma\chi^{-1}, \tag{14}$$

i.e. χ anticommutes with γ. We have thus established:

a) that χ is an antilinear operator in Lüders' sense;

b) that in the real formulation χ is the first operator we find not commuting with γ, *i.e.* not belonging to the class of usual (linear) operators.

One can throw some more light upon this last point by working out the algebra of χ and γ. Let us put

$$\eta = \gamma\chi = -\chi\gamma\,; \tag{15}$$

we get:

$$\gamma = \eta\chi = -\chi\eta\,, \tag{16}$$

$$\chi = \eta\gamma = -\gamma\eta\,, \tag{17}$$

$$\eta^2 = 1\,. \tag{18}$$

Thus we have built up a four unit algebra, containing the usual algebra of complex numbers. Every operator A acting on S' may then be written:

$$A = A_0 + A_1\gamma + A_2\chi + A_3\eta\,, \tag{19}$$

and we can identify S_2 with that subspace of S' whereon γ, χ, η operate. For the q's and χ form a complete set, whose eigenvectors will be denoted by $|j+\rangle$ if belonging to the eigenvalue $+1$ of χ, by $|j-\rangle$ if belonging to -1; S_1 is the subspace where the q's alone form a complete set, S_2 the one where χ is complete in itself. Since γ—and then η too—commutes with the q's, it must act on S_2, *q.e.d.*

Eq. (19), where A_0, A_1, A_2, A_3 act on S_1, thus commuting with γ, χ, η gives an expression for the most general operator of S'; but, as we previously saw, only the restricted class

$$A = A_0 + A_1\gamma \tag{20}$$

gives rise to operators corresponding to observables of the usual theory. We may characterize this class by its commuting with γ. This fact will be discussed further, in order to elucidate its relation to later results.

Let us now see what about equations of motion. We shall assume the

Schrödinger equation in its usual form:

$$\gamma \frac{\mathrm{d}}{\mathrm{d}t}|x\rangle = H|x\rangle \tag{21}$$

($|x\rangle$ is now a ket of S'). Keeping in mind that γ is now an operator, one can easily find:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x|x\rangle = \langle x|(H\gamma - \gamma H)|x\rangle\,, \tag{22}$$

which shows that for the sake of normalization a H of type (20) is needed. We shall put at once:

$$H = H_0 + \gamma H_1\,. \tag{23}$$

From eq. (21) the equation of motion for an operator in Heisenberg picture is easily obtained:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = [\gamma H, A]\,. \tag{24}$$

This equation differs from the usual one if, and only if, A does not commute with γ. We have no *a priori* reason to exclude such case; for « physical » operators, however, no change results in respect of complex theory. As a particular instance of the anomalous case we can write:

$$\begin{cases} \dfrac{\mathrm{d}\chi}{\mathrm{d}t} = [\gamma H, \chi] = [\gamma H_0 - H_1, \chi] = [\gamma, \chi]H_0 = 2\eta H_0\,, \\ \dfrac{\mathrm{d}\eta}{\mathrm{d}t} = [\gamma H, \eta] = [\gamma H_0 - H_1, \eta] = [\gamma, \eta]H_0 = -2\chi H_0\,, \end{cases} \tag{25}$$

showing that χ and η are not constants of motion, unless $H_0 = 0$ (*i.e.* if H is imaginary, following the « complex » terminology).

We are now in a position to discuss the time symmetry condition. Going back to Schrödinger picture, put

$$|x'\rangle = \chi|x\rangle \tag{26}$$

and evaluate $(\mathrm{d}/\mathrm{d}t)|x'\rangle$:

$$\frac{\mathrm{d}}{\mathrm{d}t}|x'\rangle = \chi\frac{\mathrm{d}}{\mathrm{d}t}|x\rangle = -\chi\gamma H|x\rangle = \gamma\chi H|x\rangle = \gamma\chi H\chi^{-1}|x'\rangle\,. \tag{27}$$

If and only if

$$H' = \chi H \chi^{-1} \tag{28}$$

does coincide with H, *i.e.* if H commutes with χ, we shall have:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{x'}\rangle = \gamma H|x'\rangle\,, \tag{29}$$

the system being thus time-symmetric. (It will be appreciated how easily this result is arrived at in the real theory).

It is worth noting that the condition for time symmetry requires $H_1 = 0$, thus being incompatible with the constancy of χ. Should we conclude that χ does not obey the general rule according to which any invariance of the equations of motion is always accompanied by the constancy of the transformation operator? The apparent paradox is easily solved by observing that in a time-symmetric system the χ-transformation does not leave the equation of motion invariant; instead, this happens if H anticommutes with χ, and then χ does remain constant in time. We find a real disagreement between invariance of the hamiltonian on one side, and of the equation of motion on the other. Generally speaking, one must consider the whole product γH, and not H alone; only if the transformation operator belongs to the class (20), *i.e.* commutes with γ, H alone will obviously suffice.

Let us now recall a result of MRT's, according to which one can always find an operator transforming states in such a way that time becomes reversed in (21). There is no difficulty in finding the same result in our theory. The operator we are looking for, let us call it U, is to anticommute with γH (as χ does in the time-symmetric case); it is shown in the Appendix that such an operator always exists if H is hermitean and commutes with γ. In the timesymmetric case χ itself may be assumed as U, but in general U can be constructed only if the eigenstates of H are known.

Anyhow, let $T(t_1 - t_0)$ be the transition operator from instant t_0 to t_1, as formally defined by:

$$\frac{\mathrm{d}}{\mathrm{d}t_1} T(t_1 - t_0) = -\gamma H T(t_1 - t_0)\,, \qquad T(0) = 1\,; \tag{30}$$

one easily obtains

$$UTU^{-1} = T^{-1}\,; \qquad T = U^{-1}T^{-1}U\,, \tag{31}$$

wherefrom:

$$\langle x|T|y\rangle = \langle x|U^{-1}T^{-1}U|y\rangle = \langle x|U^{-1}\cdot T^{-1}\cdot U|y\rangle = \langle y|U^{-1}\cdot T\cdot U|x\rangle \tag{32}$$

(remember that T is a unitary operator, acting on a real space!), *i.e.* the re-

ciprocity theorem. For a time-symmetric system (32) becomes:

$$\langle x|T|y\rangle = \langle y'|T|x'\rangle \,. \tag{33}$$

The practical value of the reciprocity theorem is tied, of course, to « reversed » states being physically realizable; to this I shall return later.

5. – Time reversal for a Fermi particle.

Before going to the final discussion of this paper, it is still needed to explain how our theory can take into account the case of systems lacking a classical analogue, *i.e.*—as far as we know—systems containing Fermi particles. For the sake of brevity, we shall consider only the simplest case, a non-relativistic particle of spin $\frac{1}{2}$.

The point we have to make clear is how to define TR for spin operators, $\sigma_1\sigma_2\sigma_3$. From the basic relations

$$\begin{cases} \sigma_1\sigma_2 = \gamma\sigma_3 \,, \\ \sigma_2\sigma_3 = \gamma\sigma_1 \,, \\ \sigma_3\sigma_1 = \gamma\delta_2 \,, \end{cases} \tag{34}$$

(where, as always, we have written γ instead of the imaginary unit) we see that if χ commutes with one of the σ's, it must commute with another and anticommute with the third; the only other possibility being that χ anticommutes with each of the σ's. In fact only the latter hypothesis may be accepted, because of the vector character of the σ's. Hence we shall have:

$$\sigma_r\chi + \chi\sigma_r = 0 \,. \tag{35}$$

Let us assume a base where σ_3 is diagonal and γ satisfies (1) and (2). The following representation will obtain:

$$\sigma_3 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} ; \quad \gamma \equiv \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

wherefrom it necessarily follows

$$\sigma_1 \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} ; \quad \sigma_2 \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(to within a linear combination of σ_1 and σ_2). Then eqs. (35) enforce the following representation of χ:

$$\chi \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

(the sign having been arbitrarily chosen). We see that χ is not hermitean, yet more resulting

$$\chi^2 = -1 . \tag{36}$$

There is no surprise in this, since we know that a twofold application of TR changes the sign of a fermion state vector; much of what we said hitherto, however, loses its validity, eq. (8) being now false.

An alternative way is also possible, or even preferable, to the one sketched above. We can maintain eq. (8) with all its consequences for the algebra of χ, η, γ; then χ will no more represent TR for the system considered in this section. This viewpoint has the advantage of being easily generalized to every system with no classical analogue; χ will keep its meaning of TR operator for « classical » degrees of freedom only; another unitary operator will still be needed for accomplishing TR on « quantum » degrees of freedom, like *e.g.* for spin, in the instance given before [14].

A further observation should be made, concerning mixed systems (*i.e.* systems containing both bosons and fermions). In such cases neither (8) nor (36) are true; as WICK, WIGHTMAN and WIGNER have shown [15] this implies the existence of a « superselection rule », at least if we assume that under two successive TR's the original state of the system is recovered. Though such an assumption appears quite obvious, I can find for it no other support than an intuitive one; I think, however, that the whole question deserves a further, careful study, no definite conclusion being possible for the present.

6. – Macro- and microscopic irreversibility: theoretical consequences.

We are now ready to undertake the discussion about the physical meaning of the theory developed so far. First, let us observe that, since χ is the TR operator, its eigenstates $|j+\rangle$, $|j-\rangle$ are time-symmetric and time-antisym-

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[14] It is easily shown that in this case full TR is effected by $\chi\sigma_2$. Of course $\chi\sigma_2 \cdot \chi\sigma_2 = -\chi^2\sigma_2^2 = -1$, this result replacing eq. (36).

[15] G. C. WICK, A. S. WIGHTMAN and E. P. WIGNER: *Phys. Rev.*, **88**, 101 (1952)

metric respectively. Moreover, it is readily seen that the eigenstates of η are $(1/\sqrt{2})(|j+\rangle + |j-\rangle)$ and $(1/\sqrt{2})(|j+\rangle - |j-\rangle)$, and these transform into each other under TR. η's eigenstates may thus be used as distinctive of time direction; *i.e.* η can be given the meaning of *time direction operator*. Hence it follows that for such states as $|j+\rangle$, $|j-\rangle$ or any other not being an eigenstate of η, *time direction is not precisely defined.*

In the above interpretation of χ, η and then of space S_2, we are faced with an experimental difficulty, the phase of state vectors being unobservable, and thus vectors of S_2 being indistinguishable from one another. We are thus compelled by experimental evidence to assume that χ and η are not observables, since the contrary assumption would imply that we are able to fix a base of S_2; vectors of that space should then be distinguishable.

On the other hand, the operational meaning assumed for χ and η requires that time direction is at our free disposal. Then we have the logical chain:

a) phase arbitrarity inhibits χ and η's being observables;

b) χ and η's being observable implies our free disposal of time direction.

A partial consequence can already be drawn: while our need of introducing space S_2 in quantum theory is due to the existence of two time directions, phase arbitrarity is due to our being unable to freely dispose of time direction itself.

One could ask whether this condition is a contingent one, or there is some kind of necessity for it in quantum theory. In my opinion the latter alternative is true. Though the whole matter is to be considered still open to discussion, I would like, as a conclusion to this paper, to give a short account of my point of view.

A careful analysis of topics related to the direction of time ([10]) shows that—at least on a macroscopic level—the unidirectionality of time derives from the second law of thermodynamics, *i.e.* from the irreversibility of macroscopic phenomena. This does not conflict with what we asserted before about two time directions: a discussion of the second law in the light of statistics (Boltzmann's famous H-theorem) reveals the « sectorial » character of time direction. That means that we are enabled to think of time intervals (of course on a cosmic scale) during which time direction was—or will be—opposite to the present one.

Anyway, in our present history, and on a macroscopic level, time does have a definite direction. This does not seem, at a first sight, to imply that the same must happen on quantum level, where no role should be played by macroscopic irreversibility; such an opinion is expressed by WIGHTMAN, BARGMANN, WIGNER ([6]). On the other hand, some recent studies on quantum theory of measurement ([8,16]) are likely to suggest irreversibility as a basic property

([16]) H. S. GREEN: *Nuovo Cimento*, **9**, 880 (1958).

for any measuring instrument ([17]); at the same time it is quite sure that in a correct foundation of quantum theory one must take into a full account the behaviour of the measuring apparatus.

We may thereby reasonably think that time irreversibility is introduced in quantum theory through the unavoidable interaction of the microscopic quantum system with the macroscopic measuring apparatus. If such a conclusion is right, it follows therefrom that *TR is by principle physically unrealizable*. If on the contrary, physicists should succeed in reversing time (then macroscopic irreversibility would prove inessential for quantum theory), for this only reason the absolute phase of state vectors would acquire a physical meaning.

A final remark is needed at this point. The whole matter of time reversal has been dealt with up to now in a rather formal fashion, and the present work makes no exception to the rule. Even our discussion about physical realizability of TR was carried through on a mainly theoretical line, neglecting any precise reference to actual physical questions. I cannot help of thinking of such a state of affairs as a provisional one; on the other hand, to take up here the question would have meant to go considerably beyond the scope of this work. We are not faced here, indeed, with a peculiarity of TR, but with a situation involving a wide part of the theory of reference changes and related laws of invariance. To all this matter, like to other topics mentioned in this paper, a further study will be devoted.

* * *

I wich to express my thanks to Prof. L. A. Radicati for his constant encouragement and for many useful discussions on this subject.

Appendix

We are able to show more than strictly necessary. In fact, it will be shown that if H is an hermitean operator commuting with γ, an infinity of unitary operators exist which commute with H and anticommute with γ.

([17]) One can understand such point by two ways. First, we can observe that the whole measuring technique on atomic or subatomic systems makes use of typically irreversible phenomena, such as impressing of photographic emulsions, gas discharges, condensing or boiling of liquids. Secondly, on a theoretical ground, we note that in order to get macroscopic effects from microscopic causes an « amplifying » device is required, consisting in some sort of « cascade » or « chain reaction », bringing the measuring instrument from a metastable state to an equilibrium one.

Let us take a base $|j+\rangle$, $|j-\rangle$ defined as follows:

$$H|j+\rangle = E_j|j+\rangle\,, \tag{A.1}$$

$$H|j-\rangle = E_j|j-\rangle\,, \tag{A.2}$$

$$\gamma|j+\rangle = |j-\rangle\,, \tag{A.3}$$

$$\gamma|j-\rangle = -|j+\rangle\,. \tag{A.4}$$

The above relations give an unambiguous determination of the base only if no one of the eigenvalues E_j of H is degenerate; in all cases, however, at least one base satisfying (A.1)–(A.4) always exists. The commutation relations with H and γ of the operator U we are looking for will be of the desired kind if U is given the form:

$$U = \sum_j \left(|j+\rangle u_j^+ \langle j+| - |j-\rangle u_j^+ \langle j-| + |j+\rangle u_j^- \langle j-| + |j-\rangle u_j^- \langle j+|\right)\,, \tag{A.5}$$

where the u_j^+, u_j^-'s are to satisfy

$$(u_j^+)^2 + (u_j^-)^2 = 1\,, \tag{A.6}$$

if U is to be unitary. One particularly simple solution obtains if we set $u_j^+ = 1$, $u_j^- = 0$. U will then be diagonal in the assumed base, and will be written:

$$U = \sum_j \left(|j+\rangle\langle j+| - |j-\rangle\langle j-|\right)\,. \tag{A.7}$$

Therefore there always exists an U commuting with H and anticommuting with γ; of course it also anticommutes with γH, as we wanted. But the converse is also true: if U anticommutes with γH it must anticommute with γ, and then commute with H. The proof is quite easy, and it will not be reported here: we only observe that in order to remove some exceptional case that could arise, it is necessary to make use of H being defined to within an additive real constant.

We are thus led to the conclusion that U may always be expressed as a product of χ and an operator of class (20). This result shows that whichever definition of TR is adopted, χ will always play an essential role, and then its properties are what really matters to know. We have thus found a further justification of our definition of TR, as χ directly occurs in it as the main object to be studied.

Note added in proof:

D. Finkelstein, J. M. Jauch and D. Speiser have recently taken into consideration an extension of the complex field of ordinary quantum mechanics, thus building up a « Quaternion Quantum Mechanics » (CERN Reports: 59-7, 59-9, 59-17). It turns out that QQM is a rather deep reformulation of « complex » quantum mechanics, but it has some attracting features, which should afford for an easier introduction of

isotropic spin, and perhaps for an explanation of the unique value of the elementary quantum of charge.

In the opinion of these authors the need for a complex or quaternion field should therefore be connected with the chargè properties of elementary particles, rather than to the direction of time. Since I have not yet fully understood the physical meaning of QQM, I cannot attempt a comparison of the two points of view: it will be an interesting task for the future.

RIASSUNTO

Si discute la particolare posizione occupata nella teoria quantistica dall'inversione del tempo, a causa dell'intervento di operatori antilineari. Questo fatto dovrebbe chiarire la funzione che hanno i numeri complessi nella teoria quantistica: si prospetta un'interpretazione in termini di una irreversibilità operativa del tempo, conseguenza dell'influenza sui sistemi quantistici degli apparati di misura, essenzialmente irreversibili per loro natura.

X-Ray Diffraction Analysis for Mixtures of Dioxane with Water Solutions of Manganese Sulphate or Cupric Sulphate.

F. Cennamo and E. Tartaglione

Istituto di Fisica Sperimentale dell'Università - Napoli

(ricevuto il 30 Marzo 1959)

Summary. — X-ray diffraction analysis shows that the opposite action of Mn^{++} and Cu^{++} on water molecules (dissociating for Mn^{++} and associating for Cu^{++}) takes place even in three-component systems containing water, each of the above mentioned ions and dioxane in different proportions.

We discuss here some interesting features exhibited by water solutions of manganese sulphate and of cupric sulphate.

1. – We can say first of all that by analysis of the Raman effect studied in the case of $MnSO_4$ solutions (1), the intensity of the band group characteristic for water at 3500 cm^{-1} has been shown to increase noticeably in presence of the Mn^{++} ion. Now this band group presents three maxima at 3200, 3435 and 3600 cm^{-1} respectively, the last two having been classified as corresponding to the symmetric and the antisymmetric frequencies of the simple water molecules, so that an increase of their intensities denotes an increased number of simple molecules and, correspondingly, a depolymerization effect by Mn^{++} on water. On the other hand, it has been noticed that the diffraction ring characteristic for water vanishes completely in presence of Mn^{++}, whence, this ring being generally understood as due to molecular associations, one can infer independently, as shown by X-ray analysis, that the Mn^{++} ion exerts a depolymerizating action on water (2).

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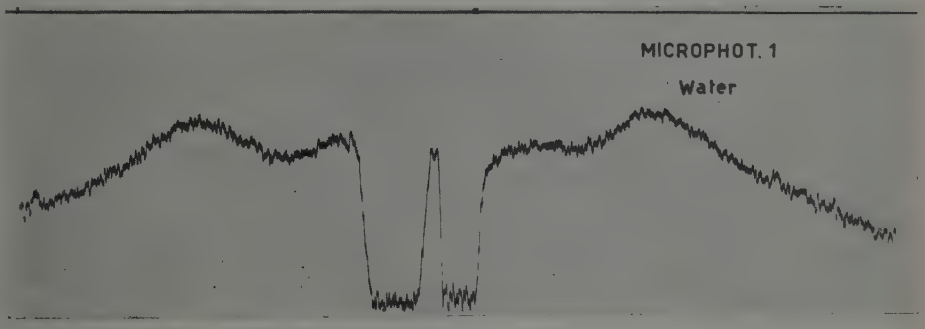
(1) F. Cennamo: *Nuovo Cimento*, **13**, 304 (1936).
(2) E. Tartaglione: *Nuovo Cimento*, **8**, 12 (1951).

Moreover, various other experiments with X-ray diffraction by water-dioxane solutions have proved the fact that water molecular associations are destroyed by dioxane at any value of the dioxane molecular concentration lower than about 15% ([3]). So we have been prompted to turn our analysis from two-component systems to the three component system formed by water, manganese sulphate and dioxane, with varying dioxane content.

By using a water solution of manganese sulphate, *i.e.* 33 g of salt in 100 cm^3 of solution and letting the volume percent of dioxane increase, we noticed that the system presents: a homogeneous appearance up to 20% of dioxane volume, two phases between 20% and 80%, *i.e.* the water solutions of manganese sulphate and of dioxane, three phases at more than 80%, solid salt being added as precipitate.

Let us note that, when the system shows an interruption of the mixing process, no trace whatever of manganese can be found by chemical analysis in the phase water-dioxane. That is, in the interval of these percentages, dioxane must be subtracting water from the solution without mixing with it. The behaviour of these systems under X-ray diffraction has been studied by the same experimental method discussed elsewhere ([4]), using the wavelength CuK_α ($\lambda = 1.54$ Å). According to the experimental results so obtained, the one-phase systems with dioxane concentrations not exceeding 20% in volume do not show any diffraction ring characteristic either for water or for dioxane or for those water-dioxane solutions corresponding to the said interval of concentrations.



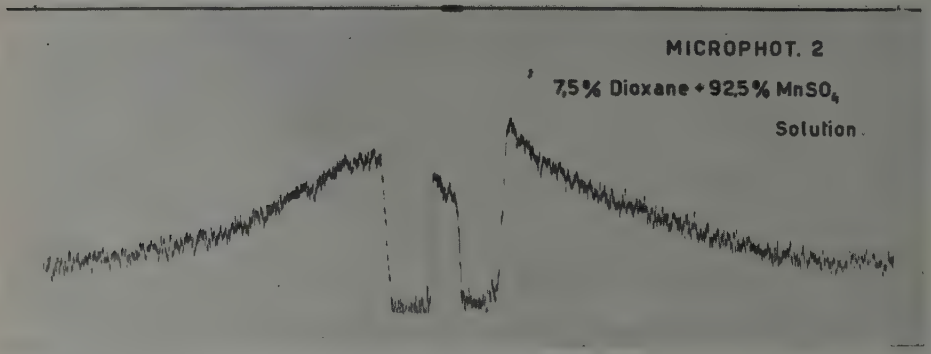
Microphotography 1.

Microphotometry 2 refers to one of the above systems, namely that at concentration 7.5% in volume of dioxane: clearly, one does not see there any trace of diffraction rings, at any characteristic position, *i.e.* that for water

([3]) F. Cennamo and E. Tartaglione: *Nuovo Cimento*, **11**, 401 (1959).
([4]) F. Cennamo: *Nuovo Cimento*, **10**, 395 (1953).

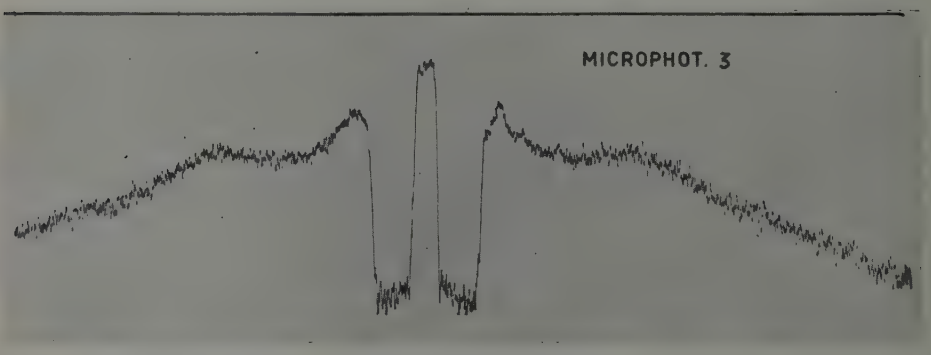
(Microph. 1) for dioxane (Microph. 4) and that for the corresponding water-dioxane solution (Microph. 3).

It seems worthwhile to point out that water-dioxane solutions in the said interval of dioxane concentrations are characterized by diffraction rings whose



Microphotography 2.

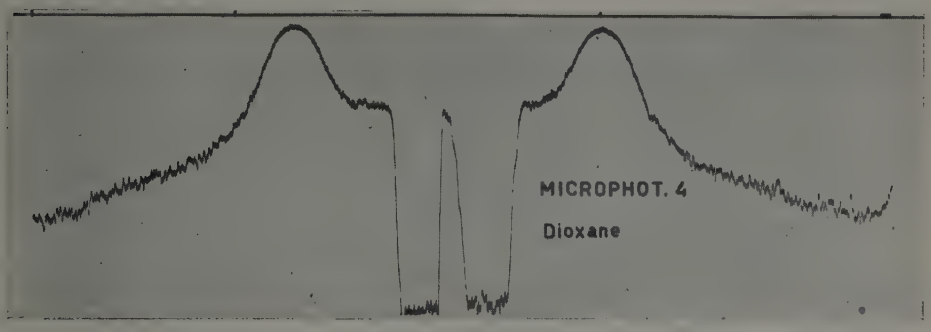
intensity and diameter both vary for each solution (like that shown in Microph. 3) and that such rings do not simply result from the two pure component rings superimposed ([3]).



Microphotography 3.

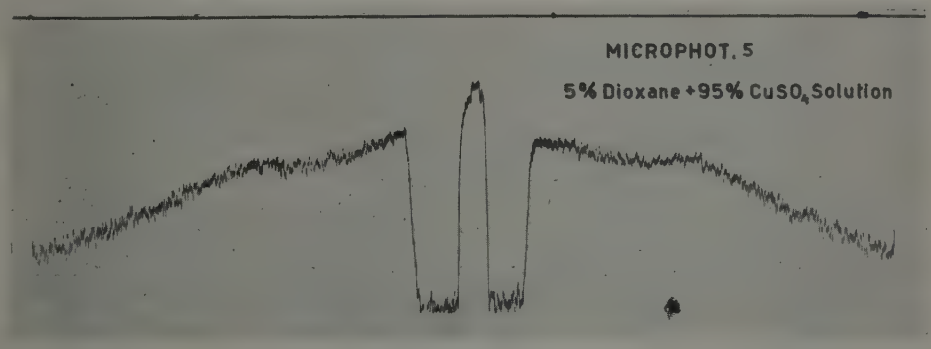
Let us now consider the two phases obtained from systems with dioxane concentrations higher than 20% in volume. At X-ray diffraction the phase water-manganese sulphate behaves like the water solution of sulphate, *i.e.* it does not show any diffraction ring. Analogously, the phase water-dioxane behaves like the water-dioxane solutions in the same concentration interval:

for each of these solutions and in the same position as for the pure dioxane, a crisp diffraction ring appears whose intensity increases with the dioxane content.



Microphotography 4.

2. – Following the same procedure with water solutions of cupric sulphate (1), by analysis of the Raman effect the intensity of the already mentioned band group characteristic for water at 3 500 cm^{-1} has been shown to decrease noticeably in presence of the Cu^{++} ion; which denotes a decreasing number of simple water molecules or, correspondingly, an increase of molecular associations. Analogous results have been observed at X-ray diffraction: the diffraction ring of water does not shift its position while showing an increased intensity; which again means an increase of associations (2).



Microphotography 5.

In order to study, as before in the case of the manganese sulphate, how the three-component system formed by cupric sulphate, water and dioxane with varying dioxane content behaves at X-ray diffraction, a saturated water solu-

tion of cupric sulphate has been used. Such a solution mixes with dioxane in any percentage; here also we have dioxane subtracting water from the solution and solid salt as precipitate, but now, differently from the manganese case, the liquid phase remains the only phase and contains all three components.

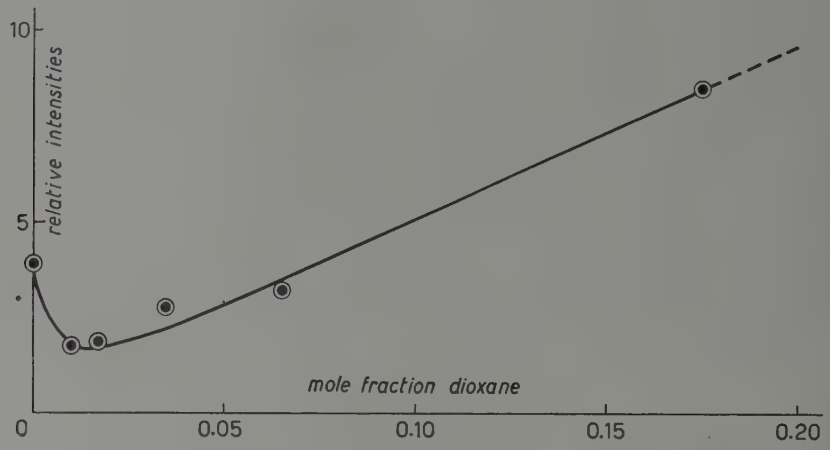


Fig. 1.

At X-ray diffraction the same experimental techniques show a diffraction ring characteristic for each of these systems, which varies with the dioxane content both in position and in intensity. (Microph. 5 shows the behaviour of one of these mixtures).

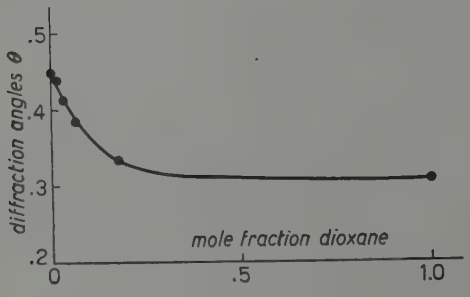


Fig. 2.

In order to make the results here considered comparable with those obtained for the water-dioxane solutions ([3]), the dioxane percentages of the system have been reckoned neglecting the cupric sulphate molecules.

Fig. 1 shows the distribution of the relative intensities, as measured at the maximum of the diffraction ring, for water and for solutions with molecular content of dioxane not exceeding 20%. Here only data relative to the said concentration interval, the only relevant for our research ([3]), are given; furthermore it is known that the intensity varies linearly at higher concentrations.

In this case also, as for water-dioxane solutions, the behaviour of the relative intensity shows a minimum though it appears shifted towards lower values of the dioxane concentration, *i.e.* 1.5% instead of 7%. On the other hand, the position of the maximum varies in the same fashion for both cases (Fig. 2).

3. – Discussion of results.

1) *For the system water-manganese sulphate-dioxane.* Dissociation of water molecules being promoted by the Mn^{++} ion, it follows that in mixtures containing water and Mn^{++} (Microph. 2) no influence is to be expected from dioxane on water, which explains why no contribution to the X-ray diffraction relative to the concentration interval considered is here evidenced. Obviously, where the manganese ion is absent one can foresee the same behaviour shown in the case of the water-dioxane solutions, that is the intensity increases with the dioxane content while the position of maximum remains fixed.

2) *For the system water-cupric sulphate-dioxane.* Association of water molecules being promoted by the Cu^{++} ion, it follows that the behaviour of dioxane in mixtures containing Cu^{++} is expected to coincide with that shown in the water-dioxane solutions, which actually, as previously seen, is in agreement with the experimental evidence, if due allowance be made for the slight shifting of the position of minimum noticed in Fig. 1; this last circumstance may be explained by supposing that associations between water and Cu^{++} are being formed out of the water-water associations pre-existing in water at its pure state.

* * *

The authors express their thanks to Prof. Antonio Carrelli, director of the Institute for Experimental Physics of the Naples University, for the means made available and the stimulating discussions held on this research.

RIASSUNTO

L'analisi alla diffrazione dei raggi X mostra che l'azione opposta del Mn^{++} e del Cu^{++} sulle molecole di acqua (azione dissociante per il Mn^{++} ed associante per il Cu^{++}) si manifesta anche nei sistemi a tre componenti contenenti acqua, ciascuno dei due ioni citati e diossano in varie proporzioni.

Dipolar Rotation Effect in Liquids.

E. Grossetti

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(ricevuto il 30 Marzo 1959)

Summary. — In this paper are reported the rotation moment values for some polar liquids (toluene, nitrobenzene, ethyl alcohol, amyl alcohol, metyl alcohol, distilled water, acetic acid) in a rotating electric field at frequencies 8.5, 12.0 and 23.0 MHz, in order to determine by a different method the viscosity coefficient values. This was accomplished by determining η with Born's formula, when the values for M of the electric moment, measured in the polarization processes according to Debye's formula, are substituted in it. For certain liquids, however (toluene, nitrobenzene, acetic acid), there is to point out that the values obtained in this way for η are noticeably different from those obtained by classical measurements, *e.g.* by the Ostwald's viscosimeter.

Rotation moments generated in polar liquids submitted in rotating electric fields of 0.325 to 8.5 MHz frequency have been determined in a previous work (1). We found that in this frequency range and in relation to the liquids used, the rotation moments obtained depend mainly on the electric conductivity and the dielectric constant, and that the effect depending on the presence of dipoles is not yet measurable. In fact the experimental values found in the previous work have been used in order to compare them to the ones given by Lampa's formula (2) (which takes into account the conductivity and the dielectric constant of the liquids used) and to prove that this formula is in agreement with the experimental data.

(1) E. Grossetti: *Nuovo Cimento*, **10**, 2 (1959).
(2) A. Lampa: *Win. Ber.*, **115** (2*a*), 1659 (1906).

In this work, on the contrary, the rotation moments have been determined for various liquids and for rotating electric fields, at higher frequencies, that is to say at 8.5, 12.0, 23.0 MHz; at such frequencies, in fact, the contribution due to the rotation, on account of the effect of the conductivity and the dielectric constant of the liquids examined, may thus be considered negligible.

The comparison between the experimental values of the rotation moments obtained and the values we can deduce from Born's formula ([3]):

$$L = \tfrac{4}{3}\pi a^3 \eta \omega (ME/KT)^2$$

gives the possibility, when the values known and those obtained through Debye's formula on the behaviour of the dielectric constant are substituted for M, of determining the values of the coefficient of viscosity η by a quite different method.

In this case, as a matter of fact, the measurement is based on a rotation process of the molecules.

The experimental values obtained for L, in relation to the different liquids,

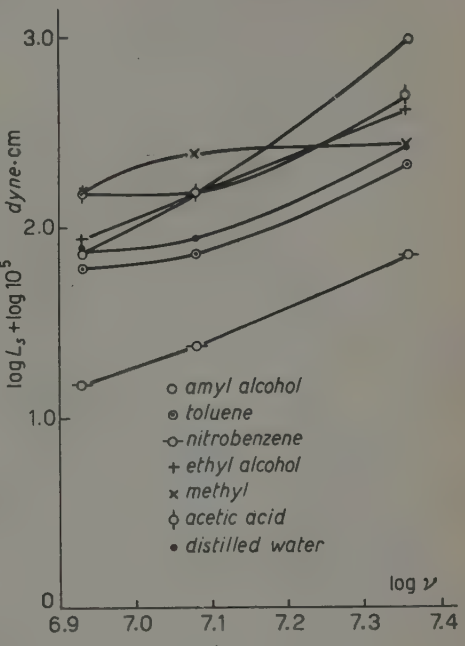

Fig. 1.

TABLE I. – *Rotation moments* dyn·cm.

	8.5 MHz		12.0 MHz		23.0 MHz	
	L_s	L_t	L_s	L_t	L_s	L_t
Toluene	$6.0 \cdot 10^{-4}$	$3{,}74 \cdot 10^{-5}$	$7.4 \cdot 10^{-4}$	$52.5 \cdot 10^{-4}$	$2.18 \cdot 10^{-3}$	$0.1 \cdot 10^{-3}$
Nitrobenzene	$1.57 \cdot 10^{-4}$	$7.20 \cdot 10^{-4}$	$0.24 \cdot 10^{-3}$	$1.02 \cdot 10^{-3}$	$0.74 \cdot 10^{-3}$	$1.93 \cdot 10^{-3}$
Ethyl alcohol	$8.70 \cdot 10^{-4}$	$7.52 \cdot 10^{-4}$	$1.57 \cdot 10^{-3}$	$1.06 \cdot 10^{-3}$	$4.17 \cdot 10^{-3}$	$2.04 \cdot 10^{-3}$
Amyl alcohol	$7.57 \cdot 10^{-4}$	$41.8 \cdot 10^{-4}$	$2.44 \cdot 10^{-3}$	$5.88 \cdot 10^{-3}$	$1.05 \cdot 10^{-2}$	$1.17 \cdot 10^{-2}$
Distilled water	$8.05 \cdot 10^{-4}$	$7.20 \cdot 10^{-4}$	$8.70 \cdot 10^{-4}$	$9.90 \cdot 10^{-4}$	$2.62 \cdot 10^{-3}$	$1.91 \cdot 10^{-3}$
Metyl alcohol	$1.61 \cdot 10^{-3}$	$0.63 \cdot 10^{-3}$	$2.53 \cdot 10^{-3}$	$0.58 \cdot 10^{-3}$	$2.81 \cdot 10^{-3}$	$1.70 \cdot 10^{-3}$
Acetic acid	$1.57 \cdot 10^{-3}$	$0.53 \cdot 10^{-3}$	$1.57 \cdot 10^{-3}$	$0.75 \cdot 10^{-3}$	$4.96 \cdot 10^{-3}$	$1.45 \cdot 10^{-3}$

([3]) M. BORN: *Zeit. f. Phys.*, **22**, 1920 (1920).

prove, in first instance, that approximately the proportionality of L in respect of ω may be considered valid. In Fig. 1 and 2 the logarithms of the rotation

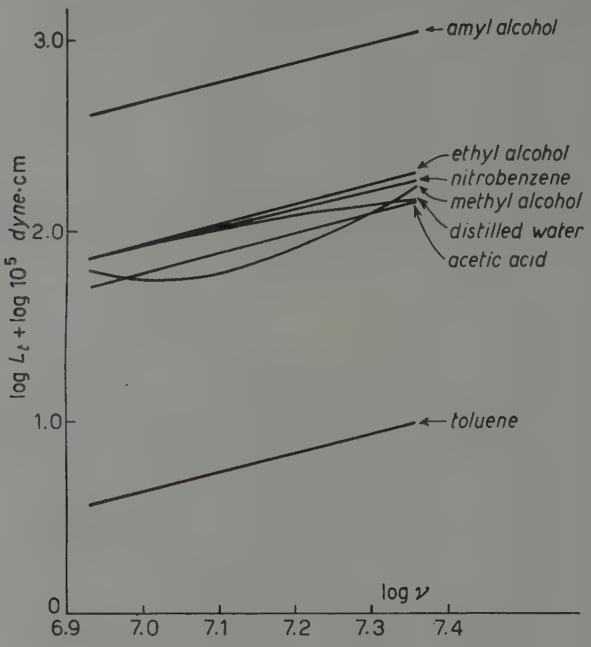

Fig. 2.

moments of the liquids are plotted against the logarithms of the frequencies, while in Fig. 3 the logarithms of the ratios between the experimental rotation moments and the theoretical ones are plotted against the logarithms of the frequencies.

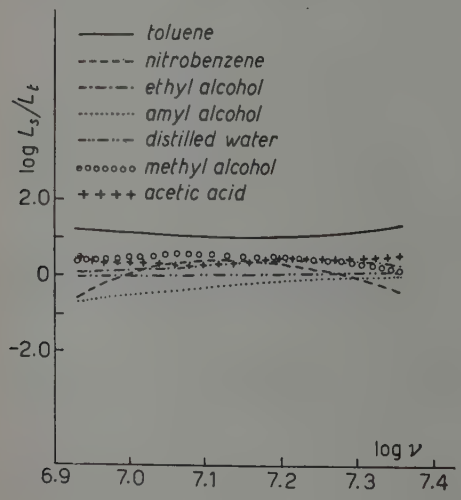

Fig. 3.

Table I, besides, shows the experimental values of the rotation moments and those theoretically calculated by means of Born's formula.

From these data it is possible also to deduce that, in this frequency range and for certain liquids, supposing that for M, which appears in the formula, the value measured in the polarization processes may be taken, the value of η thus obtained is noticeably different from that determined with Ostwald's viscosimeter.

As this result has been obtained for certain liquids, such as toluol, nitrobenzene, acetic acid, and a reason cannot yet be found to explain such differences, it is necessary that these measurements be extended to a greater number of liquids in a wider frequency range, in order to find a solution of the problem.

* * *

I express my gratitude to Prof. CARRELLI for his suggestions and for the means put at my disposal.

RIASSUNTO

In questo lavoro sono riportati i valori ottenuti per i momenti di rotazione di alcuni liquidi polari (toluolo, nitrobenzene, alcool etilico, alcool amilico, alcool metilico, acqua distillata, acido acetico) in un campo elettrico rotante alle frequenze 8.5, 12.0, 23.0 MHz, per determinare con metodo diverso i valori del coefficiente di viscosità e cioè determinando η mediante la formula di Born, quando in essa si sostituisce ad M i valori del momento elettrico misurati nei processi di polarizzazione secondo la formula di Debye. Per certi liquidi (toluolo, nitrobenzene, acido acetico) è però da notare che i valori così ottenuti per η sono notevolmente differenti da quelli che si hanno con misure classiche, per esempio con il viscosimetro di Ostwald.

K^--p Scattering at Low Energy.

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(ricevuto il 31 Marzo 1959)

Summary. — The experimental results for K^--p collision obtained by Glasser *et al.* are analyzed in a phenomenological way. One of our conclusions is that the experimental results for K^--p scattering at low energy can be explained by taking account of the effects of absorption, even if the values of phase shifts are not so large. Moreover, whether the phenomenological analysis worked out by Ascoli *et al.* is reasonable or not is discussed.

1. – Introduction.

Various approaches to $K^\pm$-p interaction have recently been done by many authors ([1-11]). The experimental results for $K^\pm$-p scattering are as follows: In the low energy region ($(20 \div 80)$ MeV), the cross-section for K^+-p scattering

([1]) L. W. Alvarez, H. Bradner, P. Falk-Vairant, J. D. Gow, A. H. Rosenfeld, F. T. Solmitz and R. D. Tripp: *Nuovo Cimento*, **5**, 1026 (1957).

([2]) F. C. Gilbert, C. E. Violet and R. S. White: *Phys. Rev.*, **103**, 1825 (1956).

([3]) W. Alles, N. N. Biswas, M. Ceccarelli and J. Crussard: *Nuovo Cimento*, **6**, 571 (1957).

([4]) E. Lohrmann, M. Nikolić, M. Schneeberger, P. Waloschek and H. Winzeler: *Nuovo Cimento*, **7**, 163 (1958).

([5]) G. Ascoli, R. D. Hill and T. S. Yoon: *Nuovo Cimento*, **9**, 813 (1958).

([6]) J. D. Jackson, D. G. Ravenhall and H. W. Wyld jr.: *Nuovo Cimento*, **9**, 834 (1958).

([7]) R. G. Glasser, N. Seeman and G. A. Snow: *Nuovo Cimento*, **9**, 1085 (1958).

([8]) C. Ceolin, N. Dallaporta and L. Taffara: *Nuovo Cimento*, **10**, 186 (1958).

([9]) K. Igi: *Progr. Theor. Phys.*, **20**, 403 (1958).

([10]) P. T. Matthews and A. Salam: *Phys. Rev.*, **110**, 565, 569, (1958).

([11]) A. Komatsuzawa, R. Sugano and Y. Nogami: *Progr. Theor. Phys.*, **21**, 151 (1959).

is of the order of 15 mb, while that for K^--p scattering is of the order of $(40 \div 60)$ mb. The purpose of this paper is to investigate the following two problems: i) how to explain the remarkable result that the latter is much larger than the former, ii) whether the phenomenological analysis worked out by ASCOLI *et al.* [5] is reasonable or not.

In K^+-p collision at low energy there is only one process $K^+ + p \to K^+ + p$, on the other hand, in K^--p collision there are the following reactions in addition to scattering:

$$K^- + p \to \Sigma^\pm + \pi^\mp$$

$$\to \Sigma^0 + \pi^0$$

$$\to \Lambda^0 + \pi^0 .$$

We think it worth-while to examine the shadow scattering due to these reactions. In Section 2 it is shown that the experimental results for K^--p scattering in low energy can be explained by taking into account the effects of absorption, even if the values of phase shifts are not so large.

ASCOLI *et al.* [5] have analyzed their experimental data for K^--p collision in terms of complex phase shifts for s-wave $\delta_{0,1} = \alpha_{0,1} + i\beta_{0,1}$ (the subscripts 0 and 1 stand for the states of the isotopic spin $I = 0$ and $I = 1$ respectively). They have assumed the simplest possible energy dependence of $\delta_{0,1}$, *viz.*,

$$k \cot \delta_{0,1} = 1/a_{0,1} , \tag{1}$$

and shown that there exist four sets of solutions for the values of $a_{0,1}$. Based on this result they have concluded that at this stage little can be inferred from the data even about the approximate amount of $I = 0$ and $I = 1$ states entering into the K^--p interaction at low energy.

We shall point out in Section 3 the following result which is derived from the assumption (1) when a (*) is pure imaginary ($a = ia'$).

i) In such an energy region as $ka' < 1$, the phase shift α is always equal to zero (in general $n\pi$).

ii) In such an energy region as $ka' > 1$, the phase shift α is always equal to $\pm\pi/2$ (in general $(2n+1)\pi/2$).

([12]) With regard to such a treatment, cfr. Y. YAMAGUCHI: *Supplement of Progr. Theor. Phys.* (to be published).

(*) For simplicity, we omit the subscripts which discriminate the isotopic spin states.

In treating the scattering problem it is difficult for us to understand such a curious behavior of α. The solution [1] see p. 360 ($a_0 = 5i \cdot 10^{-13}$ cm, $a_1 = 0$) given by ASCOLI *et al.* [5] corresponds to this case. Finally we also discuss about the other three solutions given by them.

2. – Phenomenological analysis for K^--p collision.

The experimental results obtained by GLASSER *et al.* [7] are shown in Table I. From these it may easily be expected that the effects of absorption play an important role in K^--p scattering. Let us examine these effects under

TABLE I.

Median energy of K^- (MeV)	$\sigma(K^- + p \to K^- + p)$ (mb)	$\sigma(K^- + p \to \Sigma^\pm + \pi^\mp)$ (mb)
21	49^{+32}_{-19}	49^{+30}_{-20}
47	41^{+17}_{-12}	35^{+16}_{-11}
72	30^{+13}_{-10}	13^{+10}_{-5}

the assumption that the spins of K and hyperon are respectively 0 and $\frac{1}{2}$.

We denote the reaction amplitudes for isotopic spin $I = 0$ and $I = 1$ states in $K^- + p \to \Sigma + \pi$ as A_0 and A_1 respectively, and the reaction amplitude for $I = 1$ state in $K^- + p \to \Lambda^0 + \pi^0$ as A_1', then

$$\sigma(K^- + p \to \Sigma^\pm + \pi^\mp) = (1/2)|A_1|^2 + (1/3)|A_0|^2, \tag{2}$$

$$\sigma(K^- + p \to \Sigma^0 + \pi^0) = (1/6)|A_0|^2, \tag{3}$$

and

$$\sigma(K^- + p \to \Lambda^0 + \pi^0) = (1/2)|A_1'|^2. \tag{4}$$

Considering the experimental result [1] with respect to the absorption of K^- by proton ($\Sigma^-: \Sigma^+: \Sigma^0: \Lambda^0 = 4: 2: 2: \frac{1}{2}$), we assume tentatively the branching ratio for the reactions as follows:

$$\sigma(K^- + p \to \Sigma^\pm + \pi^\mp) : \sigma(K^- + p \to \Sigma^0 + \pi^0) : \sigma(K^- + p \to \Lambda^0 + \pi^0) = 6 : 2 : \tfrac{1}{2}. \tag{5}$$ (*)

(*) With regard to this assumption, there is enough ground for controversy. Some discussion about this assumption will be done elsewhere.

Here it is necessary to remark that our assumption is not inconsistent with the experimental result ([6,7])

$$\sigma(K^- + p \to \Sigma^+ + \pi^-)/\sigma(K^- + p \to \Sigma^- + \pi^+) \simeq 1 ,$$

because the relation (5) can be interpreted as

$$\sigma(K^- + p \to \Sigma^+ + \pi^-) : \sigma(K^- + p \to \Sigma^- + \pi^+) : \sigma(K^- + p \to \Sigma^0 + \pi^0) :$$
$$: \sigma(K^- + p \to \Lambda^0 + \pi^0) = 3 : 3 : 2 : \tfrac{1}{2} .$$

It follows from the assumption that

$$|A_0|^2 : |A_1|^2 : |A_1'|^2 = 12 : 4 : 1 . \tag{6}$$

According to the experimental data ([5]), the angular distributions not only for K^--p inelastic but also for elastic scattering can be regarded as isotropic, so it may be said that s-wave interaction is mainly responsible for K^--p collision in the low energy region (*). Then A_0, A_1 and A_1' are expressed in terms of the R-matrices for s-wave, that is R_0, R_1 and R_1', respectively, and

$$\begin{cases} \sigma(K^- + p \to \Sigma^\pm + \pi^\mp) = (\pi/k^2)[(1/2)|R_1|^2 + (1/3)|R_0|^2] , \\ \sigma(K^- + p \to \Sigma^0 + \pi^0) = (\pi/k^2)(1/6)|R_0|^2 , \\ \sigma(K^- + p \to \Lambda^0 + \pi^0) = (\pi/k^2)(1/2)|R_1'|^2 . \end{cases} \tag{7}$$

Taking into account the effects of absorption, the R-matrices for isotopic spin $I=0$ and $I=1$ states in elastic scattering are expressed as $\eta_0 \exp[2i\alpha_0] - 1$ and $\eta_1 \exp[2i\alpha_1] - 1$ respectively, and the cross-sections for scattering turn out to be of the following forms

$$\sigma(K^- + p \to K^- + p) =$$
$$= (\pi/4k^2)[\eta_0^2 + \eta_1^2 + 4 - 4\eta_0 \cos 2\alpha_0 - 4\eta_1 \cos 2\alpha_1 + 2\eta_0\eta_1 \cos 2(\alpha_0 - \alpha_1)] , \tag{8}$$

$$\sigma(K^- + p \to \overline{K}^0 + n) = (\pi/4k^2)[\eta_0^2 + \eta_1^2 - 2\eta_0\eta_1 \cos 2(\alpha_0 - \alpha_1)] , \tag{9}$$

(*) In interpreting the isotropic angular distribution, there is another possibility, that is, the $p_{\frac{1}{2}}$ state in the K^--p system plays the most important role. Even then, our discussion which will be developed below is not affected in the essential point.

where

$$(10) \qquad \begin{cases} \eta_0 = \sqrt{1-|R_0|^2}\,, \\ \eta_1 = \sqrt{1-|R_1|^2-|R_1'|^2}\,. \end{cases}$$

Using the relation (6)

$$(11) \qquad \begin{cases} |R_0|^2 = (k^2/\pi)\times 2\sigma(\mathrm{K}^- + \mathrm{p} \to \Sigma^\pm + \pi^\mp)\,, \\ |R_1|^2 + |R_1'|^2 = (k^2/\pi)\times(5/6)\sigma(\mathrm{K}^- + \mathrm{p} \to \Sigma^\pm + \pi^\mp)\,. \end{cases}$$

Since we are able to get the values of η_0 and η_1 from eqs. (10) and (11) by employing the experimental values $\sigma(\mathrm{K}^- + \mathrm{p} \to \Sigma^\pm + \pi^\mp)$, the phase shifts α_0 and α_1 can generally be evaluated by eqs. (8) and (9) when $\sigma(\mathrm{K}^- + \mathrm{p} \to \mathrm{K}^- + \mathrm{p})$ and $\sigma(\mathrm{K}^- + \mathrm{p} \to \overline{\mathrm{K}}^0 + \mathrm{n})$ are given. But it will be difficult practically to obtain the reliable values of phase shifts by such a way because of the following fact. So far as the phenomena of K^--p collision in $(20 \div 80)$ MeV are concerned, η_0 calculated under our assumption turns our to be very small, and the result $\cos 2(\alpha_0 - \alpha_1)$ suffers a large change depending sensitively on the unobserved magnitude of the cross-section for charge exchange scattering. In other words, the cross-section for scattering does not depend strongly upon the value of α_0 because the magnitude of η_0 is very small. It is illustrated that in this energy region η_0 may approximately be regarded as zero.

On account of the condition $|R_0|^2 \leqslant 1$, the following relation must be satisfied,

$$\sigma(\mathrm{K}^- + \mathrm{p} \to \Sigma^\pm + \pi^\mp) \leqslant \pi/2k^2 \quad (^*).$$

The limiting values $\pi/2k^2$ are shown in Table II.

TABLE II. – *Limiting values of* $\sigma(\mathrm{K}^- + \mathrm{p} \to \Sigma^\pm + \pi^\mp)$.

Energy of K^- (MeV)	$\pi/2k^2$ (mb)
21	68.6
47	30.6
72	19.9

Since these limiting values in Table II are nearly equal to the experimental ones in Table I, let us carry on our analysis with the crude assumption in

(*) Cfr. eq. (11).

which η_0 is regarded as zero (*). Then it follows from eqs. (10) and (11) that η_1 is equal to 0.763 8, and the cross-sections for charge exchange scattering can immediately be calculated by (9). The results are shown in Table III.

TABLE III.

Energy of K^- (MeV)	$\sigma(K^- + p \to \bar{K}^0 + n)$ (mb)
21	20
47	8.9
72	5.8

α_1 can be estimated from (8) by employing the experimental data for elastic scattering. When the values of $\sigma(K^- + p \to K^- + p)$ are changed within the limits of experimental errors (cf. Table I), the allowable range for α_1 is shown in Fig. 1.

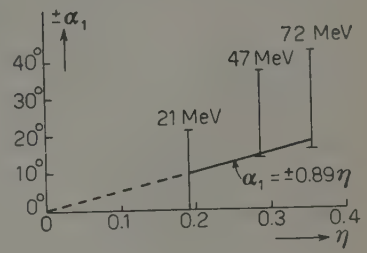

Fig. 1.

If the energy dependence of α_1 in the low energy region is expressed by the form of $\alpha_1 = c_1\eta$, we obtain $c_1 = \pm(0.89 \div 1.97)$ from Fig. 1, where η is the momentum of the K-meson in the center-of-mass system in units of m_K (K-meson mass). In order to show the importance of absorption effects, we write down in Table IV the values of $\sigma(K^- + p \to K^- + p)$ calculated by assuming $\alpha_0 = \pm\alpha_1 = \pm 0.89\eta$ without regard to the absorption effects.

TABLE IV. – *Calculated values of* $\sigma(K^- + p \to K^- + p)$.

Energy of K^- (MeV)	Our case where the absorption effects are considered ($\alpha_1 = \pm 0.89\eta$) (mb)	The case where the absorption effects are not considered	
		$\alpha_0 = \alpha_1 = \pm 0.89\eta$ (mb)	$\alpha_0 = -\alpha_1 = \pm 0.89\eta$ (mb)
21	58.3	15.6	0.45
47	29.3	15.5	0.99
72	21.1	15.4	1.47

(*) This assumption may not be used except in the case of such an energy region as this.

Although some assumptions are comprised in our estimation, it may be said from our results that the experimental results for K^--p scattering are able to be explained to some extent by taking into account the effects of absorption.

3. – Discussion.

ASCOLI *et al.* have analyzed their experimental data for K^--p collision in terms of complex phase shifts for s-wave $\delta_{0,1} = \alpha_{0,1} + i\beta_{0,1}$, where the subscripts 0 and 1 stand for the states of the isotopic spin $I = 0$ and $I = 1$ respectively. They have assumed the simplest energy dependence

$$k \cot \delta_{0,1} = 1/a_{0,1} , \tag{1}$$

and shown that there exist the following four sets of solutions for the values of $a_{0,1}$

[1] $a_0 = 5i \cdot 10^{-13}$ cm , $a_1 = 0$,

[2] $a_0 = 0$, $a_1 = (\pm 2.8 + 3.2i) \cdot 10^{-13}$ cm ,

[3] $a_0 = a_1 = (\pm 0.74 + 0.26i) \cdot 10^{-13}$ cm ,

[4] $a_0 = (\pm 0.63 + 1.4i) \cdot 10^{-13}$ cm , $a_1 = \pm 0.63 \cdot 10^{-13}$ cm .

Since any one of the four solutions can explain the experimental results both for $\sigma(K^- + p \to K^- + p)$ and for $\sigma(K^- + p \to \Sigma^{\pm} + \pi^{\mp})$, they have concluded that little can be inferred from the data at this stage about even the approximate amount of $I = 0$ and $I = 1$ states entering into the K^--p interaction at low energy.

Before some discussion about their solutions, let us state the general conclusion derived from the relation (1) when $a_{0,1}$ are pure imaginary. Since both δ and a are complex quantities, we set

$$\delta = \alpha + i\beta , \qquad ka = b + ic .$$

Then, from (1)

$$\operatorname{tg} \delta = \operatorname{tg} (\alpha + i\beta) = b + ic . \tag{12}$$

Dividing eq. (12) into the real and imaginary parts

$$\sin\alpha\cosh\beta = c\sin\alpha\sinh\beta + b\cos\alpha\cosh\beta\,, \tag{13}$$

$$\cos\alpha\sinh\beta = c\cos\alpha\cosh\beta - b\sin\alpha\sinh\beta\,, \tag{14}$$

where $0 \leqslant (\sinh\beta/\cosh\beta) \leqslant 1$ (because $\beta \geqslant 0$).

Now we restrict our discussion to the case where $b = 0$, that is, a is pure imaginary ($a = ia'$). Then

$$\sin\alpha\cosh\beta = c\sin\alpha\sinh\beta\,, \tag{13'}$$

$$\cos\alpha\sinh\beta = c\cos\alpha\cosh\beta\,. \tag{14'}$$

In these relations, whether c is larger than 1 or not is a matter of importance.

If $c < 1$, $\sin\alpha$ must be equal to zero in order to satisfy the relation (13') (because $\cosh\beta \neq c\sinh\beta$).

If $c > 1$, $\cos\alpha$ must be equal to zero in order to satisfy the relation (14') (because $\sinh\beta \neq c\cosh\beta$).

If $c = 1$, (13') and (14') cannot simultaneously be satisfied unless β is infinite.

Thus we can express our conclusion as follows: In such an energy region as $c = ka' < 1$, the phase shift α is always equal to zero (in general $n\pi$), and $\sinh\beta/\cosh\beta = c$. In such an energy region as $c = ka' > 1$, the phase shift α is always equal to $\pm\pi/2$ (in general $(2n+1)\pi/2$), and $\cosh\beta/\sinh\beta = c$. At the energy of $c = ka' = 1$, β turns out to be infinite, and the R-matrix for scattering $\exp[2i\delta] - 1$ results to be -1.

We think it difficult to understand such a curious behavior of α without suspicion. The solution [1] given by ASCOLI *et al.* [5] corresponds to this case, although it may be necessary to interpret this solution in the meaning that the real part of a_0 is very small.

We show in Table V the phase shifts calculated by eqs. (13) and (14) when

TABLE V.

Energy of K^- (MeV)	Solution [2]		Solution [3]	Solution [4]	
	α_0	α_1	$\alpha_0 = \alpha_1$	α_0	α_1
21	0	$\pm 69.8°$	$\pm 19.8°$	$\pm 26.3°$	$\pm 16.8°$
47	0	$\pm 77.1°$	$\pm 28.6°$	$\pm 51.5°$	$\pm 24.3°$
72	0	$\pm 79.7°$	$\pm 34.2°$	$\pm 63.8°$	$\pm 29.2°$

the solutions [2], [3] and [4] are adopted. The values of α_1 and α_0 in the solutions [2] and [4] respectively are too large to regard them as the suitable phase shifts at such low energies. In case of the solution [3], it can easily be seen from eq. (9) that $\sigma(K^- + p \to K^0 + n)$ is always equal to zero. By some measurement for charge exchange scattering, we shall see whether this solution may be regarded as a suitable one or not.

RIASSUNTO (*)

Si analizzano fenomenologicamente i risultati sperimentali per la collisione K^--p ottenuti da GLASSER *et al.* Una delle nostre conclusioni è che i risultati sperimentali per lo scattering K^- alle basse energie possono interpretarsi tenendo conto degli effetti dell'assorbimento, anche se i valori degli spostamenti di fase non sono grandi. Si esamina inoltre la fondatezza dell'analisi fatta da ASCOLI *et al.*

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XIII, N. 2 16 Luglio 1959

Operator Functions of the Pion Field Operator.

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(ricevuto il 4 Aprile 1959)

Summary. — We give a method for evaluating matrix elements in a bare particle representation of arbitrary operator functions of the pion field operator, with special reference to the functions arising from the equivalence transformation of the leptonic and electromagnetic couplings of nucleons.

1. – Introduction.

The well-known equivalence transformation (1), which exhibits the correspondence between the PS-PS and PS-PV pion theories, operates with non-linear functions of the pion field operator φ. It is physically important because it provides the link between reality and the Chew-Low model which employs a fixed source, PV coupling with cut-off, and ignores virtual nucleon pair creation. Furthermore, a method for dealing with non-trivial functions of φ would be required in any theory with non-linear coupling. Our present interest in the problem derives from the first point (2). When the equivalence transformation is applied to the axial vector β-decay interaction, its nucleonic factor, in the non-relativistic limit for the nucleons, transforms as follows:

$$\psi^* Q \psi \to \psi^* Q' \psi \,, \tag{1}$$

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(1) J. M. Berger, L. Foldy and R. K. Osborn: *Phys. Rev.*, **87**, 1061 (1952).

(2) G. Barton: *Phys. Rev. Lett.*, **2**, 224 (1959).

where

$$Q = \gamma_4 \gamma_5 \gamma_\mu \tau_x , \tag{2}$$

$$Q' - Q = \delta Q = Q \left\{ \frac{[\boldsymbol{\varphi} \times (\boldsymbol{\varphi} \times \boldsymbol{\tau})]_x}{\varphi^2 \tau_x} [1 - (1 + G^2 \varphi^2 / m^2)^{-\frac{1}{2}}] \right\} . \tag{3}$$

In eq. (3) we have defined

$$\varphi = (\boldsymbol{\varphi} \cdot \boldsymbol{\varphi})^{\frac{1}{2}} . \tag{4}$$

Similar expressions are obtained for instance for the isotopic vector part of the nucleon contribution to the spatial components of the electric current (the pion current being unchanged), which determines the nucleon contribution to both the Dirac and the anomalous magnetic moment ([3]).

Having applied the equivalence transformation we proceed in the framework of the Chew-Low theory, within which a non-covariant perturbation treatment of the nucleon-pion interaction is acceptable ([4]). Hence the physical nucleon state vector is profitably approximated by a combination of the bare-nucleon state with a few terms each containing only a few mesons.

We must then evaluate the matrix elements of operators of the type (3) between such states, *i.e.* in the bare particle representation. The physical consequences of our results have already been given elsewhere ([2]); for the remainder of this paper we shall concentrate on the general method for dealing with such operators.

2. – The exponential operator.

Our method is based on the matrix elements of the operators $\exp[i\lambda\varphi_j]$, where φ_j is a single neutral boson field (*one* of the components of the isotopic vector $\boldsymbol{\varphi}$), and λ is a real number.

As is well known ([5]), the field operator φ_j is not well defined unless a cut-off is imposed in momentum space. We therefore write, quantizing in the usual way in a volume V,

$$\varphi_j = V^{-\frac{1}{2}} \sum_{k<K} (1/2\omega_k)^{\frac{1}{2}} \{a_k \exp[i\underline{k} \cdot \underline{r}] + a_k^* \exp[-i\underline{k} \cdot \underline{r}]\} . \tag{5}$$

([3]) Somewhat similar considerations occur in the paper by S. Gotō and S. Machida: *Prog. Theor. Phys.*, **20**, 216 (1958).

([4]) For a covariant treatment of the equivalence transformation see J. S. R. Chisholm and G. M. Dixon: *Nuovo Cimento*, **9**, 125 (1958).

([5]) S. Albertoni and F. Duimio: *Nuovo Cimento*: **6**, 1193 (1957); and references given there.

We shall find that in our particular case the limit $K \to \infty$ is well defined and finite.

Next, we rewrite φ_j as the sum of its creation and annihilation parts in the way suggested by (5):

$$\varphi_j = A + A^*,$$
$$A = V^{-\frac{1}{2}} \sum_{k<K} (1/2\omega_k)^{\frac{1}{2}} a_k \exp[i\underline{k}\cdot\underline{r}],$$

and define for future reference the commutator

$$C(K) = [A, A^*] = V^{-1} \sum_{k<K} (1/2\omega_k). \tag{6}$$

From the theorem proved in the Appendix we have the operator identity

$$\exp[i\lambda\varphi_j] = \exp[i\lambda A + i\lambda A^*] = \exp[i\lambda A^*] \exp[i\lambda A] \exp[-\lambda^2 C/2]. \tag{7}$$

Operating on the vacuum A gives zero, whence

$$\langle 0|\exp(i\lambda\varphi_j)|0\rangle = \exp[-\lambda^2 C/2]. \tag{8}$$

Consider now the matrix elements of $\exp[i\lambda\varphi_j]$ between states specified by sets of occupation numbers $[m]$ and $[n]$ for each field oscillator with wave number $k < K$. Such a state vector is a product of state vectors for each oscillator. Define

$$x_k = \{a_k \exp[i\underline{k}\cdot\underline{r}] + a_k^* \exp[-i\underline{k}\cdot\underline{r}]\}(2\omega_k V)^{-\frac{1}{2}}. \tag{9}$$

We have, dividing and multiplying the right hand side by $\langle 0|\exp[i\lambda\varphi_j]|0\rangle$,

$$\langle [m]|\exp[i\lambda\varphi_j]|[n]\rangle = \exp[-\lambda^2 C/2] \prod_{k<K} \frac{\langle m_k|\exp[i\lambda x_k]|n_k\rangle}{\langle 0|\exp[i\lambda x_k]|0\rangle}. \tag{10}$$

In the product on the right hand side of (10) only those factors differ from unity for which either or both of m_k or n_k differ from zero.

By inspecting the form of the free meson Hamiltonian and the commutation relations satisfied by φ_j it is easy to see that, except for its phase factors, x_k is exactly analogous to the position variable of a simple harmonic oscillator in ordinary quantum mechanics. By exploiting this analogy one deduces for any given field oscillator k

$$\frac{\langle m|\exp(\mu x)|n\rangle}{\langle 0|\exp(\mu x)|0\rangle} = F(k) = \left(\frac{n!\,m!}{2^{n+m}}\right)^{\frac{1}{2}} C_\mu(n, m) P. \tag{11}$$

In (11), we have abbreviated

$$\mu = i\lambda/(V\omega_k)^{\frac{1}{2}} ; \tag{12}$$

P is the appropriate phase factor,

$$P = \exp[i(m-n)\underline{r}\cdot\underline{k}] ,$$

and we have defined

$$C_\mu(n, m) = (n!\, m!\, \pi^{\frac{1}{2}})^{-1} \exp[-\mu^2/4] \int_{-\infty}^{\infty} \exp[-x^2] H_n(x) H_m(x) \exp[\mu x]\, dx . \tag{13}$$

The general formula for the $C_\mu(n, m)$ is found to be

$$C_\mu(n, m) = C_\mu(m, n) = (2/\mu)^{m+n} \sum_{s \geq \max(m,n)}^{s \leq (m+n)} (\mu^2/2)^s [(s-m)!\,(s-n)!\,(m+n-s)!]^{-1} . \tag{14}$$

In particular,

$$\begin{cases} C_\mu(0, 0) = 1 , & C_\mu(1, 1) = 2 + \mu^2 ; \\ C_\mu(1, 0) = \mu , & C_\mu(2, 0) = \mu^2/2 . \end{cases} \tag{15}$$

Note finally that the matrix elements of integral powers of φ_j can be obtained from (10) by expanding both sides in power series in λ and equating coefficients. Thus for instance

$$\langle 0 | \varphi_j^{2n} | 0 \rangle = (2n)!\, (C/2)^n/n!$$

from which we find after some manipulation that

$$\langle 0 | (\boldsymbol{\varphi}\cdot\boldsymbol{\varphi})^n | 0 \rangle = (2n+1)(2n)!\, (C/2)^n/n! .$$

3. – The Fourier analysis method.

With equations (10), (11), and (14) we are now in a position to evaluate the matrix elements of any operator function $\mathcal{O}$ of $\boldsymbol{\varphi}$.

If $\mathcal{O}$ possesses a unique expansion in powers of φ_j, then no difficulty arises since the matrix elements of each term are obtainable as discussed at the end of the last section. There are cases however where such an expansion does not exist. Thus in our particular example, eq. (3), if $(1+G^2\varphi^2/m^2)^{-\frac{1}{2}}$ is formally written as a binomial series in ascending powers of $G\varphi/m$ and the matrix

elements taken term by term, then the resulting series in powers of G diverges, demonstrating that the result cannot be expanded in such a series. We are therefore forced to employ the more powerful methods of Fourier analysis. In the sense of generalized function theory (6) we decompose $\mathcal{O}$ into a three-dimensional Fourier integral:

$$\mathcal{O} = \int f(\boldsymbol{\alpha}) \exp[i\boldsymbol{\alpha}\cdot\boldsymbol{\varphi}]\, \mathrm{d}^3\alpha\,, \tag{16}$$

where $f(\boldsymbol{\alpha})$ is defined by

$$f(\boldsymbol{\alpha}) = (2\pi)^{-3}\int \mathcal{O}(\boldsymbol{y}) \exp[-i\boldsymbol{\alpha}\cdot\boldsymbol{y}]\, \mathrm{d}^3y\,. \tag{17}$$

Taking the desired matrix element of (16),

$$\langle [m]\,|\,\mathcal{O}(\boldsymbol{\varphi})\,|\,[n]\rangle = \int f(\boldsymbol{\alpha}) \exp[-\alpha^2 C/2]\, \mathrm{d}^3\alpha \prod_{k<K} F(k)\,. \tag{18}$$

Now use (17) and reverse the order of integration. Thus,

$$\langle [m]\,|\,\mathcal{O}\,|\,[n]\rangle = (2\pi)^{-3}\int \mathrm{d}^3y\, \mathcal{O}(\boldsymbol{y}) \int \mathrm{d}^3\alpha \exp[-\alpha^2 C/2) \exp[-i\boldsymbol{\alpha}\cdot\boldsymbol{y}] \prod_k F(k)\,. \tag{18}$$

Clearly the operator $\mathcal{O}$ is meaningful only if the right hand side of eq. (18) exists. A little thought will show that it suffices to consider the existence of the vacuum expectation value:

$$\langle 0\,|\,\mathcal{O}\,|\,0\rangle = (2\pi)^{-3}\int \mathrm{d}^3y\, \mathcal{O}(\boldsymbol{y}) \int \mathrm{d}^3\alpha \exp[-\alpha^2 C/2] \exp[-i\boldsymbol{\alpha}\cdot\boldsymbol{y}]\,. \tag{19}$$

Now,

$$\int \mathrm{d}^3\alpha \exp[-\alpha^2 C/2] \exp[-i\boldsymbol{\alpha}\cdot\boldsymbol{y}] = (4\pi/y)\int_0^\infty \alpha\, \mathrm{d}\alpha \sin(\alpha y) \exp[-\alpha^2 C/2] =$$

$$= (2\pi/C)^{\frac{3}{2}} \exp[-y^2/2C]\,,$$

so that eq. (19) becomes

$$\langle 0\,|\,\mathcal{O}\,|\,0\rangle = (2\pi)^{-3}(2\pi/C)^{\frac{3}{2}}\int \mathrm{d}^3y\, \mathcal{O}(\boldsymbol{y}) \exp(-y^2/2C)\,. \tag{20}$$

Thus the existence of (20) is the criterion for that of $\mathcal{O}(\boldsymbol{\varphi})$.

(6) M. J. LIGHTHILL: *Introduction to Fourier Analysis and Generalised Functions* (Cambridge, 1958).

4. – The no-cutoff limit.

We investigate eq. (20) in the limit $K \to \infty$. Clearly

$$C(K) = V^{-1} \sum_{k<K} \left(\frac{1}{2\omega_k} \right) = (2\pi)^{-2} \int_0^K k^2 \, \mathrm{d}k (k^2 + \mu^2)^{-\frac{1}{2}}$$

diverges quadratically in this limit:

$$C(K) \to \infty . \tag{21}$$

Hence, if $\int \mathrm{d}^3 y \, \mathcal{O}(y)$ exists, then all matrix elements of $\mathcal{O}$ vanish trivially as $K \to \infty$. Otherwise we must evaluate eq. (20) with finite K before taking the limit. If $\mathcal{O}$ has no singularities for finite y, only its asymptotic behaviour as $y \to \infty$ is relevant: if

$$O(y) \sim y^a , \tag{22}$$

then

$$\langle 0 | \mathcal{O} | 0 \rangle \sim C^{-\frac{a}{2}} \int^{\infty} y^{2+a} \exp [- y^2/2C] \, \mathrm{d}y \sim C^{a/2} . \tag{23}$$

Of course (23) is relevant only if there is no selection rule causing the matrix element to vanish identically. Thus $\langle 0 | \boldsymbol{\varphi} | 0 \rangle = 0$ but $\langle 0 | \boldsymbol{\varphi} \cdot \boldsymbol{\varphi} | 0 \rangle \to \infty$ as $K \to \infty$.

To take the example discussed earlier, eq. (3), it is clear from (23) that the second term of δQ, which has $a = -1$, vanishes in the no-cutoff limit. Thus, in this limit, for any (approximate) Chew-Low state vector, we have the exact equivalence [2]

$$\delta Q = Q \left\{ \frac{[\boldsymbol{\varphi} \times (\boldsymbol{\varphi} \times \boldsymbol{\tau})]_x}{\varphi^2 \tau_x} \right\} , \tag{24}$$

which is a slight generalisation of the result given previously [2].

* * *

This paper is based on part of an Oxford doctoral thesis written under the supervision of Dr. R. J. Blin-Stoyle. It is a pleasure to acknowledge my debt to him for proposing the problem of the equivalence transformation, and to thank Dr. C. Kacser for many enjoyable discussions.

Appendix

Theorem: Given two operators α and β whose commutator γ is a c-number,

$$[\alpha, \beta] = \gamma \,, \tag{25}$$

then

$$\exp[\alpha+\beta] = \exp[\alpha]\exp[\beta]\exp[-\gamma/2] \,, \tag{26}$$

the exponentials being defined through their power series expansions.

Proof: We make the perfectly general Ansatz:

$$\exp[(\alpha+\beta)x] = \exp[\alpha x]\exp[\beta x]Q(x) \,, \tag{27}$$

where $Q(x)$ is some operator function of the c-number x, and satisfies the condition

$$Q(0) = 1 \,. \tag{28}$$

Differentiate (27) with respect to x and substitute (27) on the left hand side. Multiply by $\exp[-\alpha x]$ from the left, use the expansion

$$\exp[-\alpha x]\beta\exp[\alpha x] = \beta - [\alpha,\beta]x + [\alpha,[\alpha,\beta]]\frac{x^2}{2!} - \ldots$$

together with (25) to find

$$(\beta - x\gamma)\exp[\beta x]Q = \beta\exp[\beta x]Q + \exp[\beta x]\frac{\mathrm{d}Q}{\mathrm{d}x} \,.$$

Now multiplying by $\exp[-\beta x]$ from the left we get

$$-x\gamma Q = \mathrm{d}Q/\mathrm{d}x \,. \tag{29}$$

This is a first order linear differential equation which together with the boundary condition (28) determines $Q(x)$ uniquely. We note that it is in fact a c-number function of x; integrating (29) subject to (28) we get

$$Q(x) = \exp[-\tfrac{1}{2}\gamma x^2] \,,$$

whence, putting $x = 1$

$$\exp[\alpha+\beta] = \exp[\alpha]\exp[\beta]\exp[-\gamma/2]$$

and by symmetry also

$$\exp[\alpha+\beta] = \exp[\beta]\exp[\alpha]\exp[+\gamma/2]\,. \tag{30}$$

A theorem similar to the above is proved by GARTENHAUS and SCHWARTZ (7) for the case $[\alpha, \beta] = \gamma \cdot \alpha$. Our method of proof is slightly different from theirs in that we do not assume that a factorization of the required forms exists.

(7) S. GARTENHAUS and C. SCHWARTZ: *Phys. Rev.*, **108**, 482 (1957).

RIASSUNTO (*)

Diamo un metodo per calcolare gli elementi di matrice in una rappresentazione in termini di particelle nude di funzioni operatrici arbitrarie dell'operatore di campo del pione, con speciale riferimento alle funzioni che nascono dalla trasformazione equivalente degli accoppiamenti leptonici ed elettromagnetici dei nucleoni.

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Scattering of Λ^0 Hyperons by Nucleons at Intermediate Energies (*) (**).

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(ricevuto il 9 Aprile 1959)

Summary. — A calculation is made of the Λ^0-nucleon scattering cross-section at 75 and 150 MeV laboratory energies, based on a phenomenological central potential with a hard core to describe the Λ^0-nucleon interaction. The potential is chosen so as to give agreement with hyperfragment data and at the same time to have reasonable properties from a meson-theoretical viewpoint. With this potential the Λ^0-nucleon cross-section is calculated numerically, assuming that scattering occurs only in states with orbital angular momentum $\leqslant 2$. The result is that the Λ^0-nucleon cross-section is approximately constant in the energy range considered. The variation is from 26 mb at 75 MeV to 21 mb at 150 MeV without a spin-orbit force, and from 34 mb at 75 MeV to 32 mb at 150 MeV with a spin-orbit force. These results are consistent with the very preliminary measurement of Crawford *et al.*

1. – Introduction.

Essentially all of the information about the strong Λ^0-nucleon (ΛN) interaction has been derived from observations of the decay of hyperfragments [1]. The interpretation of these data in terms of a two-body ΛN interaction requires the assumption that three-body forces do not play an important role in contributing to the binding energies of the Λ in hyperfragments.

(*) Work supported in part by an All-University Research Grant.

(**) This calculation was previously reported in *Bull. Am. Phys. Soc.*, **3**, 402 (1958).

(1) See, for example, R. H. Dalitz and B. W. Downs: *Phys. Rev.* **111**, 967 (1958), and references contained therein.

It is therefore important to examine the two-body ΛN potential obtained from analyses of hyperfragments to see whether it can predict the correct ΛN scattering cross-sections. It is the purpose of this work to use such a ΛN potential to calculate the ΛN cross-sections between 75 and 150 MeV laboratory energies for the incident Λ. At present only fragmentary experimental information ([2]) exists about ΛN scattering, although it is expected that the use of large hydrogen bubble chambers will rectify this situation.

2. – Description of potential.

We shall use a ΛN potential which fits the hyperfragment data and which is reasonable on theoretical grounds—a central potential of exponential form with a hard core. The existing hyperfragment data can be explained without the necessity of introducing a hard core in the ΛN potential. However, the analyses which have been made so far determine only two parameters of the ΛN potential, and therefore a hard core is not excluded. We add the hard core because, on the basis of meson theory, a hard core is reasonable in the ΛN interaction if the NN potential has one ([3,4]). The simplification of the omission of tensor forces is also quite reasonable. This has been discussed previously in some detail in reference ([4]). It should be emphasized that the shape of the ΛN potential, while not too important for interpretation of hyperfragment data, is important for calculating the ΛN cross-section at intermediate energies.

Our potential then is

$$(1) \qquad \begin{cases} V(r) = -V_0 \exp[-\alpha r], & r > D, \\ V(f) = \infty, & r < D, \end{cases}$$

where $\alpha = 2.4$ fermi^{-1} and $D = 0.4$ fermi ([5]). The value of the parameter α is fixed by the requirement that an exponential potential (in the absence of a hard core) have the same intrinsic range ([6]) as a Yukawa potential which arises from the exchange of two pions. This is the longest range potential consistent with charge independence and the assignment of isotopic spin zero for the Λ. The

([2]) F. S. CRAWFORD, M. CRESTI, M. L. GOOD. F. T. SOLMITZ, M. L. STEVENSON, and H. K. TICHO: *Phys. Rev. Lett.*, **2**, 174 (1959).

([3]) D. B. LICHTENBERG and M. ROSS: *Phys. Rev.*, **107**, 1714 (1957).

([4]) D. B. LICHTENBERG: *Nuovo Cimento*, **8**, 463 (1958).

([5]) One fermi equals 10^{-13} cm.

([6]) J. M. BLATT and V. F. WEISSKOPF: *Theoretical Nuclear Physics*, (New York, 1952) p. 56.

hard core radius D agrees with that of the nucleon-nucleon potential. The strength parameter which fits the hypertriton binding energy ([4]) has the value $V_0 = 690$ MeV when the radius of the hard core is taken to be 0.4 fermi ([7]). This is a spin-averaged value. Hyperfragment data indicate that the ΛN potential is spin-dependent and that V_0 (singlet) $>$ V_0 (triplet) ([1]). Consistent with these data is our choice of ([8])

$$V_0 \text{ (singlet)} = 2V_0 \text{ (triplet)} . \tag{2}$$

We then have V_0 (singlet) $= 790$ MeV, V_0 (triplet) $= 395$ MeV.

For the purpose of comparison we have carried out our calculations with and without a spin-orbit term in the potential. For this term we have used the isotopic-spin independent spin-orbit potential which SIGNELL and MARSHAK ([9]) have found to be a useful addition to the GARTENHAUS ([10]) potential in order to fit nucleon-nucleon (NN) scattering data up to 150 MeV. (The Gartenhaus potential alone is not able to fit the NN data.) Outside the repulsive core this potential is

$$V \text{ (spin-orbit)} = \frac{V_0'}{y} \frac{\mathrm{d}}{\mathrm{d}y} \left(\frac{\exp[-y]}{y} \right) \boldsymbol{L} \cdot \boldsymbol{S} , \tag{3}$$

where $y = r/r_0$, $r_0 = 1.07$ fermi and $V_0' = 30$ MeV ([11]).

The effect of the spin-orbit term on the binding energies of the Λ in hyperfragments has not been investigated. However, all hyperfragment calculations have assumed that the Λ is in an S-state with respect to the nucleus. For the lighter hyperfragments at least, this means that the Λ is primarily in an S-state with respect to the individual nucleons. Then, since the spin-orbit force vanishes in S-states, it is reasonable to assume that its effect on the binding energies is small.

([7]) A subsequent calculation with a better trial wave function (D. TRUONG: *Bull. Am. Phys. Soc.*, **4**, 38 (1959) indicates that V_0 should be $\sim (10 \div 20)\%$ smaller than the value given here. However, when the authors became aware of this work, calculations were already finished and were not repeated.

([8]) See ref. ([4]) and R. H. DALITZ and B. W. DOWNS: *Phys. Rev.*, **110**, 958 (1958) (and to be published) for estimates of this ratio. Our value may be too small by $\sim 20\%$.

([9]) P. S. SIGNELL and R. E. MARSHAK: *Phys. Rev.*, **109**, 1229 (1958).

([10]) S. GARTENHAUS: *Phys. Rev.*, **100**, 900 (1955).

([11]) Subsequently, P. S. SIGNELL, R. ZINN, and R. E. MARSHAK: *Phys. Rev. Lett.*, 1, 416 (1958) have found that a spin-orbit potential with different parameters fits the nucleon-nucleon scattering somewhat better. However, the improvement was not great enough to warrant repeating our calculation. See in this connection M. H. HULL, K. D. PYATT, C. R. FISCHER and G. BREIT: *Phys. Rev. Lett.* **2**, 264 (1959).

3. – Calculation and results.

We have restricted our calculations of the cross-section to laboratory energies of 75 and 150 MeV. These are below the threshold for charge-exchange scattering where the appearance of the Σ-hyperons would make our phenomenological potential considerably more complicated. At these energies we can, furthermore, safely neglect angular momentum states higher than D-waves.

For the ΛN scattering the total cross-section (at energies below the threshold for charge exchange scattering) is given by

$$\sigma = \frac{\pi}{k^2} \sum_{J,L,S} (2J + 1) \sin^2 \delta_{JLS} , \tag{4}$$

where k is the relative wave number and δ_{JLS} are the pertinent phase shifts. The phase shifts were calculated by solving the Schrödinger equation numerically. To find the cross-section at each energy, one must calculate six phase shifts without spin-orbit forces, and an additional six phase shifts when spin-orbit forces are included. These phase shifts, both with and without the spin-orbit term, are listed in Table I. The cross-sections are listed in

TABLE I. – ΛN *Scattering phase shifts (degrees).*

State	No spin-orbit		With spin-orbit term	
	ΛN phase shift 75 MeV	ΛN phase shift 150 MeV	ΛN phase shift 75 MeV	ΛN phase shift 150 MeV
$^1S_0$	31	14	31	14
$^1P_1$	19	28	19	28
$^1D_2$	2	6	2	6
$^3S_1$	1	−9	1	−9
$^3P_0$	7	10	−3	−7
$^3P_1$	7	10	1	1
$^3P_2$	7	10	16	24
$^3D_1$	1	4	−1	−2
$^3D_2$	1	4	0	1
$^3D_3$	1	4	2	8

Table II. From Table I, it can be seen that none of the phase shifts changes very much between 75 and 150 MeV—there are no resonances, for example. This in turn means that the cross-sections do not vary appreciably with energy between 75 and 150 MeV, as can be seen in Table II.

TABLE II. – ΛN *Cross-sections* (mb).

Lab. energy (MeV)	No spin-orbit	With spin-orbit
75	26	34
150	21	32

Our result, therefore, is that with the potentials described in Section 2, the ΛN cross-section is approximately constant at ~ 25 or ~ 30 mb in the energy range $(75 \div 150)$ MeV. CRAWFORD *et al.* [2] have seen only 4 ΛN elastic scatterings in the energy range $(50 \div 600)$ MeV, corresponding to an average cross-section of (40 ± 20) mb, a value consistent with our calculated result.

* * *

We gratefully acknowledge the computational assistance of Mr. JOSEPH FERRAR.

RIASSUNTO (*)

Si calcola la sezione d'urto per lo scattering Λ^0-nucleone a 75 e 150 MeV nel sistema del laboratorio basandosi su un potenziale centrale fenomenologico con un hard core per descrivere l'interazione Λ^0-nucleone. Il potenziale è scelto in modo da dare accordo coi dati sugli iperframmenti e contemporaneamente avere proprietà compatibili col punto di vista della teoria dei mesoni. Con questo potenziale si calcola numericamente la sezione d'urto Λ^0-nucleone, assumendo che lo scattering avvenga solo in stati con momento angolare orbitale ≤ 2. Risulta che la sezione d'urto Λ^0-nucleone è pressochè costante nel campo d'energia considerato. La variazione è da 26 mb a 75 MeV a 21 mb a 150 MeV senza la forza spin-orbita, e da 34 mb a 75 MeV a 32 mb a 150 MeV con la forza spin-orbita. Questi risultati si accordano con le misure del tutto preliminari di CRAWFORD *et al.*

(*) *Traduzione a cura della Redazione.*

Interaction of Antilambda Hyperons with Nucleons at Intermediate Energies (*).

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(ricevuto il 9 Aprile 1959)

Summary. — Cross-sections for the scattering and annihilation of antihyperons by nucleons are calculated at 75 and 150 MeV laboratory energies. The model of the anti-Λ^0-nucleon interaction is an attractive potential at large interparticle separations and a black absorptive hole at separations less than ~ 0.4 fermis. The cross-sections are obtained both with and without a spin-orbit term in the potential. The outer part of the potential is taken to be the same as the outer part of the Λ^0-nucleon potential required to fit the binding energies of the Λ^0 in hyperfragments. This form for the outer region follows from the assumption that this part of the potential arises from the exchange of two pions. The results indicate that about one anti-Λ^0 in 50 will interact with a nucleon in a hydrogen bubble chamber at 150 MeV the others decaying in flight. There are as yet no experimental data with which to compare the calculated cross-sections.

1. – Introduction and description of potential.

At present no experimental information exists concerning the anti-Λ^0 nucleon ($\bar{\Lambda}N$) interaction. In order to ascertain roughly what might be expected when $\bar{\Lambda}N$ collisions are observed, a calculation is made of the scattering and annihilation cross-sections at intermediate energies.

In order to observe $\bar{\Lambda}N$ collisions, one must of course first produce the $\bar{\Lambda}$'s. So far, only one Λ has been seen [1]. However, it is plausible that $\bar{\Lambda}$'s

(*) This work was previously reported in *Bull. Am. Phys. Soc.*, **3**, 402 (1958)

(1) D. J. Prowse and M. Baldo-Ceolin: *Phys. Rev. Lett.*, **1**, 179 (1958); *Nuovo Cimento*, **10**, 635 (1958).

may be produced by pions in comparable numbers to antinucleons at energies sufficiently high that the available phase space volume is large (2).

From the potential point of view, the interaction of an antibaryon with a baryon is quite analogous in a certain sense to the interaction of a nuclear particle with a complex nucleus: namely, the potential has an imaginary (absorptive) part as well as a real part. BALL and CHEW (3) have already treated the nucleon-antinucleon ($N\bar{N}$) problem in this manner to get scattering and annihilation cross-sections which are consistent with experimentally determined values (4). They get the real part of the $N\bar{N}$ potential by assuming that the $N\bar{N}$ interaction is the same as the NN interaction except that when an odd number of pions is exchanged the sign of the interaction is reversed, as prescribed by the Yukawa formalism. Corresponding to the absorptive (annihilation) part of the interaction they have imposed an ingoing-wave boundary condition at some distance of separation smaller than a certain value (5).

In our case of the $\bar{\Lambda}N$ interaction, we assume that the main contribution to the potential outside the core is due to the exchange of an even number of pions (two), since one pion cannot be emitted by the Λ without violating the conservation of isotopic spin. If this assumption is correct, we can take the outer region of the $\bar{\Lambda}N$ potential to be the same as that for the ΛN interaction described in the preceding article (6). The question of the inner region of the $\bar{\Lambda}N$ interaction is more complicated. It is recognized that the mechanism of annihilation is different for $\bar{\Lambda}N$ than for $\bar{N}N$. In particular, in a $\bar{\Lambda}N$ annihilation, at least one K-meson must be emitted. Therefore the fact that the Ball-Chew model is successful in the $\bar{N}N$ problem is no guarantee that it will work for the $\bar{\Lambda}N$ case. For example, the annihilation region may not be completely black in the $\bar{\Lambda}N$ case. However, for definiteness we have assumed the annihilation region to be a perfectly absorbing black sphere by imposing the ingoing-wave boundary condition at $r = 0.4$ fermi.

2. – Results.

In $\bar{\Lambda}N$ scattering the distortion of the outgoing wave is due both to scattering by the potential and to absorption arising from the annihilation (3,7).

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(2) B. D'ESPAGNAT and J. PRENTKI: *Nucl. Phys.*, **9**, 326 (1958); D. B. LICHTENBERG: *Midwest Conference on Theoretical Physics* (Evanston Ill., March 1959).

(3) J. S. BALL and G. F. CHEW: *Phys. Rev.*, **109**, 1385 (1958).

(4) C. A. COOMBES, B. CORK, W. GALBRAITH, G. R. LAMBERTSON and W. A. WENZEL: *Phys Rev.*, **112**, 1303 (1958).

(5) See also Z. KOBA and G. TAKEDA: *Progr. Theor. Phys.*, **19**, 269 (1958).

(6) J. S. KOVACS and D. B. LICHTENBERG: Preceding article.

(7) J. M. BLATT and V. WEISSKOPF: *Theoretical Nuclear Physics* (New York, 1952), p, 317 ff.

These effects may be expressed in terms of (real) scattering phase shifts δ_{JLS} and factors of potential barrier penetration to the region where annihilation takes place. The barrier is that of an effective potential, including the effect of centrifugal forces. The cross-sections are:

$$(1) \qquad \sigma_{sc} = \frac{\pi}{2k^2} \sum_{JLS} (2J+1) \left[1 - \frac{T_{JLS}}{2} - (1 - T_{JLS})^{\frac{1}{2}} \cos 2\delta_{JLS} \right],$$

$$(2) \qquad \sigma_{an} = \frac{\pi}{4k^2} \sum_{JLS} (2J+1) T_{JLS},$$

where T_{JLS} are the penetration factors. These penetration factors (evaluated in the WKB approximation as in reference [3]) are listed in Table I.

TABLE I. – *Barrier penetration factor.*

State	No spin-orbit		With spin-orbit	
	75 MeV (Lab)	150 MeV (Lab)	75 MeV (Lab)	150 MeV (Lab)
$^1S_0$	1	1	1	1
$^1P_1$	0.42	0.44	0.42	0.44
$^1D_1$	0	0	0	0
$^3S_1$	1	1	1	1
$^3P_0$	0.20	0.30	0	0.06
$^3P_1$	0.20	0.30	0	0.12
$^3P_2$	0.20	0.30	1	1
$^3D_1$	0	0	0	0
$^3D_2$	0	0	0	0
$^3D_3$	0	0	0	0.05

In a few cases the T_{JLS} were evaluated numerically. (These cases are given to two significant figures in the table). In the other cases, it was apparent that the T_{JLS} were very nearly zero or one, and were approximated by those values.

For the phase shifts appearing in formula (1), we used the same phase shifts as were calculated in reference [6] for the ΛN case. These are not strictly the phase shifts which should be used, since the potentials are not the same in the inner region. However, note from the formula (1) that in the case $T_{JLS} = 1$, the phase shift δ_{JLS} drops out. In the opposite extreme $T_{JLS} = 0$, the wave function does not penetrate the barrier and is sensitive only to the outer region of the potential, which is the same as in the ΛN case. From Table I it can be seen that when spin-orbit forces are included, only the transmission coefficient in the $^1P_1$ state (T_{110}) is significantly different from zero, and this state has a relatively small statistical weight. Therefore, the approx-

imation of using the ΛN scattering phase shifts in the $\bar{\Lambda}$N problem is a good one. Without spin-orbit forces, the situation is much less favorable, since all the triplet P state penetration factors are neither 1 nor 0. The calculatied $\bar{\Lambda}$N cross-section appear in Table II.

TABLE II. – $\bar{\Lambda}$N *cross-sections* (mb).

Lab energy MeV	No spin-orbit			With spin-orbit		
	σ_{sc}	σ_{an}	σ_{tot}	σ_{sc}	σ_{an}	σ_{tot}
75	50	62	112	90	90	180
150	34	35	69	51	49	100

3. – Discussion.

A major question concerning this calculation is: how realistic is the $\bar{\Lambda}$N potential we have used? If the outer region of the potential is in fact due to pion exchange, then taking the $\bar{\Lambda}$N potential the same as the ΛN potential in this region is reasonable. This is not the case if the potential is due to the exchange of a single K-meson since then it will be repulsive when the ΛN potential is attractive.

In the inner region, our assumption that there is a black hole for annihilation is tenuous. However, since K-meson interactions are strong (as shown by the large $\bar{\text{K}}$-proton cross-section), and since the phase space available for annihilation is large, there is some hope this assumption may not be too bad.

Another difficulty is that our $\bar{\Lambda}$N scattering and annihilation cross-sections appear to be somewhat sensitive to the exact radius assumed for the annihilation boundary. In the $\bar{\text{N}}$N case calculated by BALL and CHEW, the cross-sections were less sensitive to this boundary, The difference comes about as follows: The assumption is made that, if the interacting particles overcome the centrifugal barrier in any state and find themselves in a real potential well, then they will annihilate independently of the radius of the annihilation boundary. In the $\bar{\text{N}}$N case, the real potential is deep enough in many states to overcome the centrifugal barrier at inter-particle separations greater than ~ 0.4 fermi, the assumed upper limit for the annihilation region. This is not true in the $\bar{\Lambda}$N case, where the annihilation region itself overcomes the centrifugal barrier in most states. Note, however, that since the barriers are thick and the penetration coefficients are very nearly 1 or 0 even in the $\bar{\Lambda}$N case, our results are not too sensitive to small changes in the annihilation boundary. A more detailed discussion of this question is given in reference [3]. Although the sensitivity to the annihilation region means that the calculated $\bar{\Lambda}$N cross-sections may not be very accurate, it indicates that if the outer part of the

potential turns out to be substantially correct, $\bar{\Lambda}N$ collision experiments may yield detailed information about the annihilation region.

It may be of interest to compare the cross sections obtained here with those obtained using a similar ingoing wave boundary condition but with the outer part of the $\bar{\Lambda}N$ potential obtained from Gell-Mann's global symmetry condition ([8]). In the latter case, the $\bar{\Lambda}N$ total cross-section σ is given by ([9])

$$\sigma = \tfrac{1}{2}\sigma_{\bar{p}p} + \tfrac{1}{2}\sigma_{\bar{n}p}\,, \tag{3}$$

where $\sigma_{\bar{p}p}$ and $\sigma_{\bar{n}p}$ are the total antiproton-proton and antineutron-proton cross-sections respectively. BALL and CHEW calculate $\frac{1}{2}\sigma_{\bar{p}p}+\frac{1}{2}\sigma_{\bar{n}p} \approx 150$ mb at 140 MeV. Our value of $\sigma_{\bar{\Lambda}N} \approx 100$ mb at 150 MeV ([10]) is in disagreement with this by $\sim 50\%$. If the Ball-Chew NN potential is correct, this means that our phenomenological $\bar{\Lambda}N$ potential differs in the outer region from the potential predicted by global symmetry.

From the magnitude of the $\bar{\Lambda}N$ cross-section calculated here, we can estimate the mean free path for a $\bar{\Lambda}$ interacting in a hydrogen bubble chamber. We can compare this length with the mean distance that the $\bar{\Lambda}$ will travel before decaying. It turns out that at 150 MeV (with spin-orbit forces) only 1 Λ in ~ 50 will interact in hydrogen—the others will decay in flight. The situation is slightly less favorable without spin-orbit forces. Thus, the scattering and annihilation experiments will be difficult to do.

([8]) M. GELL-MANN: *Phys. Rev.*, **105**, 1296 (1957).

([9]) D. B. LICHTENBERG: *Phys. Rev.* **113**, 1309 (1959).

([10]) The cross-section with spin-orbit forces is the pertinent one for this comparison, since the $\bar{N}N$ cross-section was calculated with spin-orbit forces.

RIASSUNTO (*)

Si calcolano le sezioni d'urto per lo scattering e l'annichilamento degli antiiperoni su nucleoni a 75 e 150 MeV d'energia nel sistema del laboratorio. Il modello dell'interazione anti-Λ^0-nucleone è un potenziale attrattivo con ampie separazioni fra le particelle e una buca nera assorbente per le separazioni inferiori a ~ 0.4 fermi. Le sezioni d'urto si ottengono con e senza termine spin-orbita nel potenziale. Si considera la parte esterna del potenziale uguale alla parte esterna del potenziale Λ^0-nucleone richiesta per soddisfare le energie di legame del Λ^0 negli iperframmenti. Tale forma della regione esterna segue dall'assunzione che questa parte del potenziale derivi dallo scambio di due pioni. I risultati indicano che circa 1 anti-Λ^0 su 50 interagisce con un nucleone in una camera a bolle a idrogeno a 150 MeV, decadendo gli altri in volo. Non si hanno finora dati sperimentali con cui confrontare le sezioni d'urto calcolate.

(*) *Traduzione a cura della Redazione.*

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The Inelastic Scattering of Elementary Particles.

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Summary. — The requirements of unitarity and causality are used to obtain a convenient set of real constant parameters for the phenomenological description of low energy elementary particle scattering. It is assumed that an arbitrary number of channels are open but that there are just two particles in each channel. This discussion is a direct generalization of effective range theory. We also derive from this point of view the Breit-Wigner and Chew-Low formulas. Unitarity is then used to relate the parameters, below threshold for one or more of the channels, to the larger number of parameters required when all channels are open. Finally these considerations are applied to the K^--nucleon system.

1. – Introduction.

In this paper we consider the phenomenological description of low energy elementary particle scattering. Of particular interest is the scattering of K^--mesons, and of Σ and Λ hyperons, on nucleons. Below threshold for additional π-meson production, the former involves the three channel system consisting of K^-N, $\Lambda\pi$ and $\Sigma\pi$ states, the latter is a two channel system consisting of ΛN and ΣN states. (We are also interested in the single channel system of K-nucleon and π-nucleon scattering, below threshold for π-meson production.)

These problems of inelastic scattering in several channels are closely analogous to nuclear reactions, but we find there are some special features, which do not appear to have been discussed in connection with the nuclear problem.

In particular, the inelastic processes in nuclear reactions are treated as resonance scattering. The « ordinary » scattering (by which we mean any reaction which cannot be pictured as proceeding through the formation of a

semi-stable compound state), is correctly interpreted as « potential » scattering from the surface of the nucleus and is purely elastic. In elementary particle systems, in which the basic interactions give rise to a change in the nature of the particles, the « ordinary » scattering is also largely inelastic (1). One of our main problems is to find a convenient set of parameters for a phenomenological description of this « ordinary » inelastic scattering, which incorporates the general requirements of unitarity, causality and time reversal invariance.

To this end expressions for the cross sections are set up in the next section, which clearly separate the kinematical factors (momentum dependence) due to the flux and density of states. The requirement of unitarity is also expressed in a kinematical way, and completely determines the imaginary part of the inverse of the scattering amplitude.

In Section **3** we consider the energy dependence of the real part of the scattering amplitude, which follows from the principle of causality. Combined with the results of Section **2**, this leads to a generalization of effective range theory to inelastic processes, which supplies the answer to the question posed above. We also obtain the Chew-Low (2) type expansion and give a derivation in this context of the Breit-Wigner one level formula.

In Section **4** we show how the principle of unitarity continues to operate below threshold for one or more of the channels of the system. In this way the single scattering length, which determines the elastic scattering at an energy for which only one channel is open, is related to the three or more parameters required at a higher energy, which is above threshold for additional channels.

In Section **5** these ideas are applied to the K^--nucleon system.

2. – Unitarity and Im T^{-1}.

Let us consider a system of n channels, with two particles in each channel. Let E be the total energy in the centre of mass system and E_1, E_2, ..., E_n be the threshold energies for the different channels which, without loss of generality, we label in order of increasing magnitude. For simplicity we assume that E_1, ..., E_n lie close together compared with the rest masses of the particles. In this case there is a range of values of E ($> E_n$) for which all channels are open and yet the channel momenta are all small compared with masses of the particles involved. (Equivalently all wave lengths are large compared to the range of the interaction.) We refer to this as the low energy region, and

(1) A well known example is photo-pion production where the s-wave electric dipole scattering is ordinary scattering, while the p-wave magnetic dipole proceeds through the $(\frac{3}{2}, \frac{3}{2})$ resonance. G. F. CHEW and F. LOW: *Phys. Rev.* **101**, 1579 (1956).

(2) G. F. CHEW and F. LOW: *Phys. Rev.*, **101**, 1570 (1956).

assume that in this region the entire interaction is through the S-wave. The generalization to include higher momenta is part of the standard routine of nuclear physics.

The total scattering cross-sections are determined by

$$\sigma_{if} = \frac{4\pi}{F_i} |M_{if}|^2 \varrho_f , \tag{2.1}$$

where the initial flux, F_i, and the final density of states, ϱ_f, are defined to include the energy factor $(2\varepsilon)^{-\frac{1}{2}}$ for each of the particles in the initial and final states. Then F, M and ϱ are each covariant expressions ([3]), and in particular the matrix elements M_{if} are the vacuum expectation values of Heisenberg operators, whose analytic properties have been extensively discussed in connection with dispersion relations.

In the centre of mass frame

$$\begin{cases} \varrho_f = \dfrac{p_f}{E}\,\theta(E - E_f), & \theta(E) = 1 \quad E > 0 , \\ & \quad\;\; = 0 \quad E < 0 , \\ F_i = E p_i , \end{cases} \tag{2.2}$$

where p_i and p_f are the relative momentum in the initial and final states respectively. If we now define $T_{if} = M_{if}/E$

$$\sigma_{if} = \frac{4\pi}{p_i} |T_{if}|^2 p_f \theta(E - E_f) . \tag{2.3}$$

Evidently for a system with only one channel, T is related to the s-wave phase shift:

$$T = \frac{e^{i\delta} \sin\delta}{p} = \frac{1}{p(\cot\delta - i)} . \tag{2.4}$$

The corresponding unitary S-matrix is ([1])

$$S_{if} = \delta_{if} + 2i p_i^{\frac{1}{2}} T_{if} p_f^{\frac{1}{2}} . \tag{2.5}$$

([3]) C. MØLLER: *Det. Kgl. Dan. Vid. Selskab. Mat.-Fys.*, **23**, no. 1 (1945).

([4]) The appearance of the $p^{\frac{1}{2}}$ factors may be understood as follows. From time dependent theory the matrix

$$S = \exp\left[-i\int H\,\mathrm{d}t\right],$$

is unitary. The T matrix introduced above is defined by

$$S = 1 + 2i\delta(E - E_f)T .$$

The unitarity of the S-matrix implies that it can be written in terms of a hermitian matrix A in the form

$$S = \frac{p^{-\frac{1}{2}} A p^{-\frac{1}{2}} + i}{p^{-\frac{1}{2}} A p^{-\frac{1}{2}} - i}, \tag{2.6}$$

where

$$(p^{-\frac{1}{2}})_{if} = \delta_{if} p_f^{-\frac{1}{2}} .$$

From (2.5) it then follows that, (if the matrix T has no zeros),

$$T_{if}^{-1} = A_{if} - i\delta_{if} p_f . \tag{2.7}$$

Formulas (2.3) and (2.7) exhibit all the kinematic dependence of the scattering amplitude, by which we mean the momentum dependence arising from general considerations, such as flux, phase space and unitarity. It is very important that in terms of T^{-1} the requirements of unitarity can be expressed in this simple kinematical form.

The dynamics of the system, the properties which depend on the specific interaction, are contained in the matrix A. If the theory is invariant under time reversal S, and hence A, are symmetric. Therefore A is a real symmetric matrix and thus supplies the minimum number of real parameters (functions of the energy) for the description of all the possible s-wave scattering processes of the system. (A is the inverse of the Wigner R matrix).

3. – Causality and the real part of T.

To get some further restrictions on A we now use the general principle of causality to give information about T. First let us relate the real and imaginary parts of T and T^{-1}. If we put

$$T = B + iY , \tag{3.1}$$

These are matrices both in channel space and momentum space. The product in momentum space of two such T matrices for S-state scattering is

$$\delta(E' - E_p)\,\delta(E_p - E'')(AB)_{p'p''} = A_{p'p} B_{pp''} p^2 \frac{\mathrm{d}p}{\mathrm{d}E}\,\delta(E' - E'') = A_{p'p} p B_{pp''} \delta(E' - E'').$$

Equivalently one may introduce

$$T'_{if} = p_i^{\frac{1}{2}} T_{if} p_f^{\frac{1}{2}} ,$$

which is a matrix in channel space only and satisfies (2.5).

then, from (2.7) and the relation

$$TT^{-1} = 1 , \tag{3.2}$$

it follows that

$$\begin{cases} BA - Yp = 1 , \\ Bp + YA = 0 . \end{cases} \tag{3.3}$$

Hence

$$A = B^{-1} + pA^{-1}p , \tag{3.4}$$

$$A^{-1} = B + YB^{-1}Y . \tag{3.5}$$

From this it is clear that A depends only on B. We also get the approximate solution

$$A^{-1} \simeq B , \tag{3.6}$$

and hence

$$T^{-1} \simeq B^{-1} - ip , \tag{3.7}$$

or

$$T = \frac{A^{-1}}{1 - i\sqrt{p}A^{-1}\sqrt{p}} \simeq \frac{B}{1 - i\sqrt{p}B\sqrt{p}} . \tag{3.8}$$

Since T is the S-wave part of a covariant expression it depends only on [5]

$$(k + p)^2 = E^2 , \tag{3.9}$$

where k, p are the individual four-momenta of the particles in any channel. From causality, the real part of T is an analytic function of E in the upper half plane. On the real axis it has poles at the « bound states » of the system,

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[5] In general T can be expressed in terms of two independent scalars, which may be taken to be

$$\left(\frac{p - p'}{2}\right)^2 = \Delta^2 \qquad \text{(momentum transfer)} ,$$

and

$$(p + k)^2 = E ,$$

but since it does not depend on the angle (momentum transfer) it can be expressed entirely in terms of E.

which lie in the energy region below threshold for scattering in any channel, and branch points at thresholds. The residues of the poles are the products of the appropriate renormalized coupling constants, which relate the two particle states in the channels to the bound state. Thus RlT can be written

$$RlT_{if}(E) = B_{if}(E) = \sum_{\lambda} \frac{g_i^{(\lambda)} g_f^{(\lambda)}}{E - E_\lambda} + \int_{E_1}^{\infty} \frac{f_{if}(E')}{E - E'} \, \mathrm{d}E' . \tag{3.10}$$

3·1. *Effective range theory.* – For energies E near to E_n, (the threshold energy at which all the channels of the system become open) we can expand B about E_n

$$B(E) \simeq B(E_n) + (E - E_n)B'(E^n) + \ldots . \tag{3.11}$$

Thus

$$B^{-1}(E) \simeq B^{-1}(E_n)[1 + (E - E_n)B^{-1}(E_n)B'(E_n)] \equiv a^{-1} + (E - E_n)r + \ldots . \tag{3.12}$$

Substituting into (3.7) we have the approximate expression

$$T^{-1} = [a^{-1} + (E - E_n)r + \ldots] - ip . \tag{3.13}$$

If the system has only one channel then at energies near threshold

$$E - E_n = p^2/2M , \tag{3.14}$$

where M is the reduced mass in the channel. From (2.4) and (2.7), (3.13) is just

$$p(\cot \delta - i) = \left[\frac{1}{a} + \frac{p^2}{2M} r + \ldots\right] - ip , \tag{3.15}$$

so that the matrices a and r are the direct generalizations of the scattering lengths and effective ranges to the case of inelastic scattering. In the approximation in which the « effective range » term is neglected, the scattering in an n-channel system is described in terms of a real constant symmetric matrix a^{-1}, which thus depends on

$$\tfrac{1}{2}n(n+1)$$

parameters, the generalized scattering lengths of the system. This is the

simplest set of phenomenological parameters for describing the « ordinary » inelastic scattering, in the sense defined in the introduction (6).

Note that the behaviour of cross-sections very close to threshold is determined by the kinematical factors arising from the flux F and density of states ϱ. Thus by (2.3)

$$\sigma_{if} \sim \frac{p_f}{p_i} . \tag{3.16}$$

In the scattering length approximation the remaining momentum dependence is just that required by unitarity, as expressed in (2.7)

3·2. *The Breit-Wigner formula.* – If one of the bound state levels, E_λ, is very close to threshold, one may approximate to B by

$$B_{if}(E) = \frac{g_i^{(\lambda)} g_f^{(\lambda)}}{E - E_\lambda} . \tag{3.17}$$

A similar approximate expression for B may be obtained if the expression $f(E')$ is sharply peaked about an energy E_0 in the low energy region. It may then be approximated by a δ-function

$$f_{if} = g_{if} \delta(E - E_0) . \tag{3.18}$$

If we further assume that the matrix g_{if} may be written

$$g_{if} = y_i y_f \tag{3.19}$$

we again have for B an approximate expression of the form (7) (3.17)

$$B \simeq \frac{y_i y_f}{E - E_0} . \tag{3.20}$$

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(6) These parameters are alternatives to the eigen-phase shifts and mixing parameters and avoid the necessity of transforming the physical states to a mixed representation which is awkward when different momenta are associated with different channels. See for example BLATT and BIEDENHARN: *Rev. Mod. Phys.*, **24**, 258 (1952). Y. YAMAGUCHI: *Prog. Theor. Phys.* (to be published).

(7) Note that this is a poor approximation to B near E_0, since it violates the requirements of unitarity. These are complicated in general for T, but may be seen by considering the case of a single channel system, when

$$pB = \sin 2\delta(E) .$$

This is zero at resonance and can never be greater than one. However since we later substitute into (3.8), unitarity is certainly satisfied in the final expression (3.21).

The physical significance of (3.18) is that the system can form a semi-stable state of energy E_0, and (3.19) implies that it lives long enough for the « decay amplitude » y_f to be independent of the « formation amplitude » y_i. If there is more than one channel then, in this approximation, B has no inverse and we use the form (3.8) which leads directly to the Breit-Wigner level formula in terms of the reduced widths, y,

$$T_{if} = \frac{y_i y_f}{(E - E_0) + i \sum y_s^2 p_s} . \tag{3.21}$$

This again satisfies all the requirements of unitarity.

By means of (3.21) an n-channel system can be defined in terms of $n+1$ parameters $(E_0, y_1 \ldots y_n)$. The reduction compared with the scattering length approximation arises from our assumption (3.19).

3·3. *The Chew-Low formula.* – If there is only one channel and a single bound state E_B near to E_1, then we may expand $B(E)$ about E_B,

$$B(E) \simeq \frac{g^2}{E - E_B} + R(E_B) . \tag{3.22}$$

Putting

$$E - E_B = w^* \tag{3.23}$$

and inverting, on the assumption that $w^* R$ is small, one obtains

$$B^{-1}(E) = \frac{w^*}{g^2}\left(1 - \frac{w^* R}{g^2}\right) . \tag{3.24}$$

Substituting into (3.7), this leads to

$$\frac{g^2}{w^*} T^{-1} \simeq \left[1 - \frac{w^* R}{g^2}\right] - \frac{i g^2 p}{w^*} . \tag{3.25}$$

Using (2.4), this is an identity for the imaginary part, (from unitarity), and for the real part gives

$$\frac{g^2 p \cdot \cot \delta}{w^*} = 1 - \frac{w^* R}{g^2} , \tag{3.26}$$

which is just the Chew-Low form ([2]).

In Sections 3·1 and 3·2 we have been concerned with purely phenomenological parameters. The Chew-Low formula shows how parameters closely related to the scattering length can be derived from rather general features of a particular theory. Full use of causality, unitarity and crossing symmetry in the dispersion relations gives further information about the parameters R ([8]).

4. – Unitarity below threshold and cusps.

In the discussion of unitarity given in Section 2, it was tacitly assumed that the energy E was above threshold for all the channels ($E > E^n$). We now consider the implications of the unitarity requirement when the energy is below threshold for one or more of the channels. The discussion is carried out in terms of a three channel system, since this is the most complicated case of immediate physical interest and suffices to illustrate all the general points.

Let m_1 and m_2 be the rest masses of the particles in a particular channel. Then the relative momentum in the channel, corresponding to energy E, is

$$(4.1) \quad p = [(E - m_1 - m_2 + i\varepsilon)(E - m_1 + m_2)(E + m_1 - m_2)(E + m_1 + m_2)]^{\frac{1}{2}}/2E .$$

The small negative imaginary part in the mass serves to determine the correct continuation of k, when the energy is below threshold for the channel

$$(4.2) \quad E < E_t = m_1 + m_2;$$

it has the effect of replacing p by $i|p|$ when p^2 is negative; below threshold ([9])

$$(4.3) \quad p = i|p^2|^{\frac{1}{2}} .$$

Suppose we have a three channel system labelled $\alpha = 1, 2, 3$ in order to increase threshold values. Consider the T-matrix at an energy E such that

$$(4.4) \quad E_2 < E < E_3 .$$

([8]) G. F. Chew, M. L. Goldberger, F. E. Low and Y. Nambu: *Phys. Rev.*, **106**, 1337 (1957); P. K. Roy: *Phys. Rev. Letters*, **2**, 364 (1959).

([9]) In terms of the component of the wave function in the channel, this correctly replaces the outgoing wave by a falling exponential

$$\exp[ipr] \to \exp[-|p|r] .$$

Since we have taken the matrix A to be an analytic function of E, the only change in (2.7) is the replacement (4.3) in channel 3. Thus, in this region, the inverse T-matrix can be written

$$[T^{(3)}]^{-1} = \begin{pmatrix} a - ip_2 & h & g \\ h & b - ip_2 & f \\ g & f & c' \end{pmatrix} \tag{4.5}$$

where

$$c' = c + p_3 , \tag{4.6}$$

and all the other parameters are real. The (3×3) S-matrix, $S^{(3)}$, derived from this by (2.5) is no longer unitary. However there is no reason why it should be, since all cross-sections for processes into channel 3 are zero, (as of course they must be, physically), owing to the vanishing of the θ-functions in (2.3). The physical principle of unitarity (or conservation of probability) now requires the unitarity of $S^{(2)}$, the leading two by two minor of $S^{(3)}$, which refers to the open channels. It is easy to check that this is given correctly by (4.5), as we show in the next paragraph.

The leading, two by two, minor of $T^{(3)}$ is

$$\left\{ \begin{aligned} T^{(2)} &= \frac{1}{D} \begin{pmatrix} (b - ip_2)c' - f^2 , & hc - gf \\ hc - gf , & (a - ip_1)c' - g^2 \end{pmatrix} \\ &\equiv t^{(2)}/D , \end{aligned} \right. \tag{4.7}$$

where

$$D = \det T^{(3)} . \tag{4.8}$$

To verify that $S^{(2)}$, (related to $T^{(2)}$ by (2.5)), is unitary, we invert $T^{(2)}$ and show that it has the form (2.7). This follows since

$$\det t^{(2)} = c'D . \tag{4.9}$$

Using (4.9) it is easy to show that

$$\left\{ \begin{aligned} [T^2]^{-1} &= \begin{pmatrix} a - (g^2/c') - ip_1 & h - (gf/c') \\ h - (gf/c') & b - (f^2/c') - ip_2 \end{pmatrix} \\ &\equiv A^{(2)} - ip , \end{aligned} \right. \tag{4.10}$$

which has the required form and implies the unitarity of $S^{(2)}$.

If the energy is lowered below the threshold for channel 2,

$$E < E_2 ,$$

a similar argument leads to the relation analogous to (4.10):

$$[T^{(1)}]^{-1} = \frac{a - h^2c' - g^2b' + 2fgh}{b'c' - f^2} - ip_1 , \tag{4.11}$$

where

$$b' = b + k_2 . \tag{4.12}$$

This is of the form (2.4) and shows how the single real phase shift, which describes the elastic scattering in this energy region, is related to the six parameters, which are required for the complete scattering matrix, at energies where all three channels are open [10].

The non-analytic dependence of σ_{if} on the momentum in channel $-n$, k_n, which is defined explicitly above, gives rise to the cusps in $\sigma_{if}(E)$ at the corresponding threshold [11],

$$E = E_n .$$

5. – The K-nucleon system.

The considerations presented above arose in connection with the analysis of K^--nucleon scattering. This system has three channels for the i-spin 1 states and two channels for the i-spin zero states (since $\Lambda\pi$ states are pure i-spin 1).

A preliminary analysis has been made by DALITZ [12] on the basis of complex scattering lengths which force the many channel system approxi-

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[10] Similar considerations apply to the Breit-Wigner formula. In the energy region, for example, $E_2 < E < E_3$, $T^{(2)}$ satisfies the requirements of unitarity. The effective position of the resonance is given by

$$E_0' = E_0 + |p_3| y_3^2 .$$

[11] R. H. CAPPS and W. G. HOLLADAY: *Phys. Rev.*, **99**, 931 (1955); R. K. ADAIR: *Phys. Rev.*, **111**, 632 (1958); A. BAZ and N. O'KUN: *Journ. Exp. Theor. Phys.*, **35**, 526 (1959).

[12] R. H. DALITZ: *Intern. Conf. on High Energy Physics, CERN* (Geneva 1958), p. 187.

For a system with just two channels, our real scattering length approximation is to take

$$A = \begin{pmatrix} a & f \\ f & b \end{pmatrix},$$

so that the elastic scattering amplitude is

$$[T_{11}]^{-1} = [a - f^2/(b - ip_2)] - ip_1 .$$

The square bracket would be taken as the single (complex, constant) scattering in Dalitz' treatment.

mately into the one channel mould. However, the parameters of such a theory are not adequate, even for the present limited data, as all absorption is treated simply as an attenuation of the K^--meson flux and no expressions can be derived for the branching ratios between different inelastic channels which involve interference effects between the iso-spin states. Within this framework one cannot discuss, for example, the observed strong energy dependence of the Σ^+/Σ^- ratio in the $(0 \div 20)$ MeV region.

The energy spectrum of elastic and charge exchange scattering for the K^--p interaction shows a considerable amount of structure ([13]) and it has been suggested by the present authors ([14]) that there may be a resonance in one of the isotopic spin states. If the resonance is in the i-spin 1 state the scattering amplitudes, $T^{(1)}$, in this particular iso-spin state can be described by the conventional one-level formula (3.21) with four parameters

$$E_0, \quad y_K, \quad y_\Lambda, \quad y_\Sigma$$

in an obvious notation.

In addition to the resonance scattering it would appear from the data that there is a considerable amount of ordinary scattering in the other iso-spin state. This is suggested by the fat « tails » on the resonance humps, and the smallness of the ratio of charge exchange to elastic scattering. The iso-spin zero amplitudes $T^{(0)}$ can then be described by (2.7) with A a constant real symmetric

TABLE I.

	$f(0)$	$f(1)$
$K^- + p \to p + K^-$	$\frac{1}{2}$	$\frac{1}{2}$
$\to n + K^0$	$\frac{1}{2}$	$-\frac{1}{2}$
$\to \Sigma^- + \pi^-$	$1/\sqrt{6}$	$-\frac{1}{2}$
$\to \Sigma^- + \pi^+$	$1/\sqrt{6}$	$\frac{1}{2}$
$\to \Sigma^0 + \pi^0$	$1/\sqrt{6}$	0
$\to \Lambda^0 + \pi^0$	0	$1/\sqrt{2}$
$K^- + n \to n + K^-$	0	1
$\to \Sigma^- + \pi^0$	0	$1/\sqrt{2}$
$\to \Sigma^0 + \pi^-$	0	$1/\sqrt{2}$
$\to \Lambda^0 + \pi^-$	0	1

The coefficients of the $I=0$ and $I=1$ amplitude in the various K^--nucleon processes. The ten K^0 processes are related to those given above by charge symmetry.

([13]) See report by M. F. KAPLON: *Intern. Conf. on High Energy Physics, CERN* (Geneva, 1958), p. 171.

([14]) P. T. MATTHEW and A. SALAM: *Phys. Rev. Lett.*, **2**, 226 (1959).

matrix,

$$A = \begin{pmatrix} A_{\mathrm{K}} & A_{\mathrm{K}\Sigma} \\ A_{\mathrm{K}\Sigma} & A_{\Sigma} \end{pmatrix}$$

in accordance with Section 3·1.

The complete amplitudes are given by

$$T_{if} = f_{if}(0)\, T_{i_f}^{(0)} + f_{if}(1)\, T_{if}^{(1)} ,$$

where $f(r)$ are the iso-spin factors given in Table I.

With the seven parameters introduced above it should be possible to give a complete description of all the twenty K^--nucleon interactions in the $(0 \div 100)$ MeV/c region.

RIASSUNTO (*)

Si utilizzano le esigenze di unitarietà e causalità per ottenere una opportuna serie di parametri costanti per la descrizione fenomenologica dello scattering delle particelle elementari di bassa energia. Si assume che un numero arbitrario di canali sia aperto ma che in ogni canale si trovino solo due particelle. Questa discussione è una generalizzazione diretta della teoria del range effettivo. Da questo punto di vista deriviamo pure le formule di Breit-Wigner e di Chew-Low. Si utilizza poi l'unitarietà per mettere in relazione i parametri, sotto la soglia, per uno o più canali al maggior numero di parametri richiesto quando tutti i canali sono aperti. Finalmente si applicano queste considerazioni al sistema K^--nucleone.

(*) *Traduzione a cura della Redazione.*

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A Note on the Pauli Trasformation.

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Summary. — Definitions are given of a « simple particle of spin $\frac{1}{2}$ » and of a « theory of the Heisenberg type » (1). It is shown that in a theory of the Heisenberg type simple particles must have mass 0. It is further shown that in order to obtain particles of mass $m \neq 0$ from such a theory it is necessary that one should be able to construct at least two asymptotic spinors from the spinor of the Heisenberg type theory and that these two asymptotic spinors must not transform in the same way under the Pauli transformation. One particular case: two asymptotic spinors transforming respectively as $\exp[\pm i\gamma_5\alpha]$ is discussed as an illustration.

1. – Definitions.

In the present note I want to discuss some aspects of the group represented by the unitary transformation

$$S_\alpha \psi(x) S_\alpha^+ = \exp[i\gamma_5\alpha]\psi(x), \tag{1}$$

in which S_α is a unitary operator depending on a continuous real parameter α and ψ is the operator of the spinor field. For the sake of simplicity I shall assume that ψ is a Majorana spinor, so that in Majorana's gauge ($\gamma_\nu^+ = \gamma_\nu$, $\gamma_i^* = \gamma_i$, $\gamma_4^* = -\gamma_4$; $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, $\gamma_5^+ = \gamma_5$, $\gamma_5^* = -\gamma_5$, (+) indicating Hermitian conjugation and (*) complex conjugation) one has $\psi = \psi^+$.

(1) W. HEISENBERG: preprint to be published in *Zeits. f. Naturfor.*; R. ASCOLI and W. HEISENBERG: *Zeits. f. Naturfor.*, **12***a*, 177 (1957).

A theory of the Heisenberg type is a theory in which all physical information can be derived from the matrix elements of a finite number g ($=1, 2,...$) of Majorana fields. Only the case $g=1$ will be considered in this paper and it will be evident that the case of higher g can be discussed in a completely similar manner.

The connection of the group (1) with the vanishing of the masses of Dirac or Majorana particles has been known for some time [2], but it does not appear that a sufficiently general proof has so far been given.

In the present note it will be shown that a theory of the Heisenberg type — which will be specified more accurately at the end of this section—cannot give « simple particles » of spin $\frac{1}{2}$ and mass $m \neq 0$. A simple particle is defined in the following manner:

In a theory which is invariant under the proper inhomogeneous Lorentz group it is always possible to define an energy momentum four vector I_μ (as the generator of infinitesimal translations) as well as an angular momentum operator J_i ($i=1, 2, 3$, as the generator of infinitesimal rotations) and one has quite generally $[J_i I_0]=0$. For a particle with mass $m \neq 0$ one can therefore always define two states χ_j by means of the equations

$$I_0 \chi_j = m\chi_j\,, \qquad I_i \chi_j = 0\,, \qquad J_3 \chi_j = \tfrac{1}{2} j N_j \tag{2}$$

and one has $j=\pm 1$. The particle is now called simple if the most general solution of the first two equations (2) can be written as a superposition of the two χ_j, *viz.*:

$$\chi = \sum_{j=\pm 1} \alpha_j \chi_j\,, \tag{3}$$

i.e. if the two χ_j form a complete set of eigenstates for the particle. In the sense of this definition a Majorana particle of mass $\neq 0$ is simple, but an electron is not (it is degenerate) since the definition of its state requires the specification of its charge.

It also immediately follows from this definition that the states of a simple particle of mass $\neq 0$ form an irreducible representation of the rotation group, while the states of a degenerate particle form a reducible representation. One can use this fact to generalize the definition of simple massive particles to the case of arbitrary spin: a particle the states of which form an irreducible representation of the rotation group is called simple.

The definition of a theory of the Heisenberg type of degree g is given by the following requirements:

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[2] B. Touschek: *Nuovo Cimento*, **5**, 755 (1957).

1) Schur's lemma holds for a spinor composed of g Majorana fields: any operator which commutes with all the g fields must then be a multiple of the identity operator. The actual Heisenberg theory has $g = 2$ [1].

2) The theory is invariant under (1) for $g = 1$ and under the full Pauli group [3] for arbitrary g. Here we shall only treat the first case. From the invariance of the theory under (1) we can then deduce the existence of a Hermitian operator N defined as the infinitesimal generator of S_α ($\delta S_\alpha = i\delta\alpha N$) [4] wich—because of (1)—satisfies the commutation relations

$$[N\psi(x)] = -\gamma_5\psi(x)\,. \tag{4}$$

3) The theory is invariant under an infinitesimal proper Lorentz transformation L which transforms the co-ordinates x_μ into $x'_\mu = x_\mu + \varepsilon_{\mu\nu}x_\nu$. Under this transformation the ψ transform as

$$L\psi(P)L^+ = (1 + \tfrac{1}{8}\,\varepsilon_{\mu\nu}[\gamma_\mu\gamma_\nu])\psi(P)\,, \tag{5}$$

where P indicates a point in four space. J_3 is then given as the generator, of an infinitesimal rotation ($\varepsilon_{12} = -\varepsilon_{21} = \varphi$) and one has with $L = 1 + i\varphi J_3$:

$$[J_3\psi(x)] = -i\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} + \frac{i}{2}\gamma_1\gamma_2\right)\psi(x)\,. \tag{6}$$

From equations (4) and (6) one immediately obtains by the application of Schur's lemma

$$[J_3 N] = \beta I\,, \tag{7}$$

where I is the identity operator and β is a real constant.

4) The theory admits one reflection R, which without loss of generality we can assume to have the form

$$R\psi(x)R^+ = i\gamma_4\psi(\tilde{x})\,. \qquad (\tilde{x} = (-\boldsymbol{x}, t))\,: \tag{8}$$

An application of Schur's lemma then gives $N + RNR^+ = \gamma I$, where γ is a real c number. Since if N satisfies (4) also $N - (\gamma/2)I$ will satisfy (4) we may

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[3] W. Pauli: *Nuovo Cimento*, **6**, 204 (1957); W. Thirring: *Phys. Rev.*, **111**, 986 (1958); B. Touschek: *Nuovo Cimento*, **3**, 181 (1958).

[4] Here and in the following we assume a positive metric in Hilbert space. It appears that the argument is not essentially different for a metric which is not positive definite.

normalize N in such a way, that

$$RNR^{+} = -N . \tag{9}$$

This normalization immediately rids us of the constant β in equation (7). For it follows from the properties of the rotation group that one must have

$$RJ_3R^{+} = J_3, \tag{10}$$

i.e. that the reflection cannot change the eigenvalues of J_3. Applying (9) and (10) to equation (7) one immediately derives $\beta = 0$. Finally it is postulated that

5) There exists a non degenerate vacuum state χ_0, defined as the eigenstate to the eigenvalue zero of all the ten generators of the proper inhomogeneous Lorentz group. It then follows from (9) that

$$N\chi_0 = 0 . \tag{11}$$

A theory which satisfies the postulates 1) to 5) will be called a primitive theory of degree g. It must be emphasized that such a theory differs in many respects from the theory proposed by Heisenberg. But it is the purpose of this note to illustrate, rather than to develop, some features of this theory with the purpose of studying the consequences of the invariance properties described by equation (1).

2. – The mass theorem.

The following theorem can now be proven. A primitive theory of degree one cannot give simple particles of mass $m \neq 0$ and of spin $\frac{1}{2}$.

For, if there were such a simple particle, we could immediately deduce from equation (7) that

$$N\chi_j = n(j)\chi_j , \tag{12}$$

where $n(j)$ is a c-number. For, since N commutes with the generators of the proper inhomogeneous Lorentz group $N\chi_j$ must still be an eigenstate of the operators of equation (2) and it must belong to the same eigenvalues. Using equation (8) it then follows that

$$N(R\chi_j) = -n(j)(R\chi_j) .$$

But, since R commutes with, J_3, it follows $n(j) = -n(j) = 0$. This is a contradiction, since we can show that a particle of spin $\frac{1}{2}$ must have odd eigenvalues of N. This is seen by observing that with $\psi_\pm = \frac{1}{2}(1 \pm \gamma_5)\psi$ one has

$$[N\psi_\pm] = \mp \psi_\pm \tag{13}$$

so that the $\psi_\pm$ increase or diminish the eigenvalues of n by unity. It follows that a product $\psi_+^m \psi^n$ of operators applied to the vacuum χ_0 gives a state to the eigenvalue $n - m$ of N. But for a particle of spin $\frac{1}{2}$ only those products for which $m+n$ is odd contribute. It follows that also $n - m$ must be odd (5).

This argument shows that in a theory of the type considered here the massive particles of spin $\frac{1}{2}$ must necessarily be degenerate. If a realistic theory could be made out of the assumptions made here one fact could be written to its credit: the only stable massive particles known in nature are the electron and the proton: both are degenerate. There exists one non-degenerate particle of spin $\frac{1}{2}$ and it has mass 0. In the sense of the wider definition of simple particles given in the previous section this situation holds with one remarkable exception (the π^0-meson) also for unstable particles and bosons: the π^0-meson is the only simple massive particle so far known.

The mass degeneracy between particles and antiparticles is not relieved by the breakdown of parity, it certainly holds if the theory is invariant under CP.

3. – Mass and degeneracy.

In this Section I want to study the case of degenerate particles of spin $\frac{1}{2}$ with the purpose of understanding some necessary properties of such a degeneracy. The simplest form of degeneracy will be assumed, in which the state of the degenerate particle is described by χ_{nj} ($n = \pm 1$ and the χ_{nj} are solutions of the equations

$$I_0\chi_{nj} = m\chi_{nj}\,, \quad I_i\chi_{nj} = 0\,, \quad J_3\chi_{nj} = \tfrac{1}{2}j\chi_{nj}\,, \quad N\chi_{nj} = n\chi_{nj}\,. \tag{14}$$

It is obvious that these four states require the definition of two real « in » fields $\varphi_r = \varphi_r^+$ ($r = 1, 2$) which by some procedure which we shall illustrate later have to be derived from ψ by a limiting process. The first problem to resolve is then the following: in what way must the φ_s transform if ψ is subjected to the transformation (1)? Since the φ_s represent free fields, which

(5) This conclusion is not valid if a degenerate vacuum is assumed — or a vacuum which is not an eigenstate of N.

under any transformation of the interpolating field ψ must undergo a canonical transformation, it is clear that the transformations which the φ_s undergo under (1) must form a subgroup of the group defined by

$$\delta\varphi_s = (A_{st} + i\gamma_5 S_{st})\varphi_t \,, \tag{15}$$

which is isomorphic to the Pauli groups I and II [3]. A is a real antisymmetric infinitesimal 2×2 matrix and S is antisymmetric. It follows that by using a set I, ϱ_1, ϱ_2, ϱ_3 of Pauli matrices one may put

$$A = i\zeta_2 \varrho_2 \,, \qquad S = \zeta_1 \varrho_1 + \zeta_3 \varrho_3 + \eta I \,, \tag{16}$$

where ζ_i and η are infinitesimal real numbers and the matrices ϱ_i and I act on the indices of the « internal » space (s, t). Since it has be assumed that the particle described by the two Majorana fields φ_s has a mass, it follows that there must exist at least one of the two symmetrical matrices X, Y, which by means of

$$M = m\varphi_s \lambda_4 (X_{st} + i\gamma_5 Y_{st})\varphi_t \,, \tag{17}$$

can be used to define a mass operator. Such an operator has to be invariant under the proper Lorentz group. The symmetry properties of X and Y are obtained by remembering that the two free Majorana fields are quantized with anticommutation relations and that therefore $\gamma_4 X$ as well as $i\gamma_4\gamma_5 Y$ must be antisymmetrical with respect to the interchange of all indices (*i.e.* spinor indices and internal space index s). Since γ_4 as well as $i\gamma_4\gamma_5$ are skew symmetric it follows that X as well as Y must be symmetrical. Since it is further required that both φ_s satisfy the Klein Gordon equation

$$(\Box - m^2)\varphi_s = 0 \tag{18}$$

it follows from (17) that one must have

$$X^2 + Y^2 = 1 \,, \qquad [XY] = 0 \,. \tag{19}$$

In order to explore the transformations which the φ undergo under (1), one has to find that subgroup of (15) which leaves the mass operator (17) invariant. for since the theory must be invariant under the transformation (1) also its asymptotic equations must be invariant under this transformation. It can then be easily shown [3] that the admissible subgroup of (15) is defined by the conditions

$$[A, X] = [A, Y] = \{S, X\} = \{S, Y\} = 0 \tag{20}$$

from which one can immediately deduce that there can be no mass unless $\eta = 0$. It also follows from (20) that no generality is lost by putting $Y = 0$. For the conditions (20) require that the variants of the X and Y term vanish separately and no mixing of these two terms can occur. For X we can now choose a representation in terms of Pauli matrices:

$$X = x_1 \varrho_1 + x_3 \varrho_3 + yI \,, \tag{21}$$

in which x_1 and x_3 and y are real numbers, the choice being dictated by the fact that X must be symmetrical. It is then readily seen that from $[A, X] = 0$ it follows that

$$\zeta_2 x_1 = \zeta_2 x_3 = 0$$

and further from $\{S, X\} = 0$ that

$$y\zeta_1 = y\zeta_3 = 0 \,, \qquad x_1\zeta_1 = x_3\zeta_3 = 0 \,.$$

This leads to the result which is shown in the following table:

(22)

	ζ_1	ζ_2	ζ_3	η
x_1	0	0	+	0
x_3	+	0	0	0
y	0	+	0	0

The + signs indicate the parameter combinations for which the definition of a non vanishing mass term is possible. It is seen that every admissible choice of the mass term admits exactly a one parameter group. This is a special case of the result obtained in reference [3], according to which an ensemble of n Majorana fields allows a subgroup of the extended Pauli group with $\frac{1}{2}n(n-1)$ parameters.

The combination $y = 1$, $\zeta_2 =$ arbitrary corresponds to the traditional representation of the electron. For if one sets $\varphi = \varphi_1 - i\varphi_2$ and therefore $\varphi^+ = \varphi_1 + i\varphi_2$ one sees that (15) becomes

$$\delta\varphi = -i\zeta_2\varphi \,, \qquad \delta\varphi^+ = i\zeta_2\varphi^+ \tag{23}$$

and this is the form of the infinitesimal gauge transformation used in electrodynamics. The same way of introducing a complex spinor gives a mass term $M = m\bar{\varphi}\varphi$ (with $\bar{\varphi} = \varphi^+\gamma_4$) *i.e.* the normal mass term of Dirac's theory.

The remaining two possibilities are completely equivalent to one another. Taking in particular the combination represented by the first line of table (22), it is seen that the subgroup of (15) compatible with the existence of a mass term becomes

$$\delta\boldsymbol{\varphi} = i\alpha\gamma_5\varrho_3\boldsymbol{\varphi}\,; \qquad \boldsymbol{\varphi} = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \tag{24}$$

and the mass term itself

$$M = m\boldsymbol{\varphi}\gamma_4\varrho_1\boldsymbol{\varphi}\,. \tag{25}$$

The equivalent of the Dirac equation in this case becomes

$$\begin{cases} \eth\varphi_1 + m\varphi_2 = 0 \\ \eth\varphi_2 + m\varphi_1 = 0 \end{cases} \qquad \eth = \gamma_\mu\frac{\partial}{\partial x_\mu}, \tag{26}$$

and it is seen that the two fields appear as coupled via the mass term. Equations (26) are of the form introduced by GÜRSEY [6]. They have the particular property that they are meaningful also for « two component spinors », *i.e.* if the fields φ_s would have the property $\varphi_1 = \gamma_5\varphi_1$; $\varphi_2 = -\gamma_5\varphi_2$. We shall see in Section 5 that also this case can be made completely equivalent to the « traditional » treatment of the Dirac equation represented by the last line of the table (22).

4. – Application to the Heisenberg-type theory.

It follows from the mass theorem that if a Heisenberg type theory can give massive particles of spin $\frac{1}{2}$ it is necessary that one can derive at least two asymptotic fields φ_1, φ_2 from the interpolating Majorana spinor ψ. One of the asymptotical fields φ_1, say, can be chosen to satisfy the condition

$$\psi \to \varphi_1\,, \tag{27}$$

where the arrow indicates « weak convergence » and it is obvious that under the transformation (1) one must have

$$S_\alpha\varphi_1(x)S_\alpha^+ = \exp\left[i\gamma_5\alpha\right]\varphi_1(x)\,. \tag{28}$$

[6] F. GÜRSEY: *Nuovo Cimento*, **7**, 411 (1958).

It then follows from the considerations of the last section and in particular from an inspection of the table (22) that the second asymptotic field must transform under (1) as

$$S_\alpha \varphi_2(x) S_\alpha^+ = \exp[-i\gamma_5\alpha]\varphi_2(x) . \tag{29}$$

This poses immediately the question: Is it possible to form from the interpolating field ψ an asymptotic field φ_2 which under (1) transforms like (29)? It is immediately obvious that there can be no linear relation between φ_2 and ψ. For it follows from the invariance under the proper Lorentz group that the only linear operations which we are allowed to carry out on ψ are multiplication by a factor $c_1+c_2\gamma_5$ (with c_1 and c_2 c-numbers). But this multiplication cannot change the sign of the exponent in equation (1). It follows that one has to search for the functions of ψ which asymptotically tend towards φ_2 among the non-linear combinations of ψ. This search is facilitated by observing that the quantity

$$a_\nu(x) = -i\psi\gamma_5\gamma_\nu\psi \tag{30}$$

is an invariant under the transformation (1). Indeed it is the only non vanishing bilinear invariant that can be formed from a Majorana field. Using this axial vector invariant we can construct a real spinor $\hat{\psi}$ by means of

$$\hat{\psi} = i\lambda^3\gamma_5\gamma_\nu\psi(x)a_\nu(x) . \tag{31}$$

That $\hat{\psi}$ is indeed real follows from the fact (32) has real space- and imaginary time components and that the same holds for the combination $i\gamma_5\gamma_\nu$. It is also obvious that $\hat{\psi}$ transforms like a spinor under the proper Lorentz group. The parameter λ has the dimensions of a length and it is introduced in equation (31) because $a_\nu(x)$ has the dimensions of a density. Also, because of the factor $\gamma_5\gamma_\nu$ in (31) and because of the invariance of (30) under the transformation (1) one has

$$S_\alpha \hat{\psi}(x) S_\alpha^+ = \exp[-i\gamma_5\alpha]\hat{\psi}(x) \tag{32}$$

so that in the sense of the preceding argument it is permissible to put asymptotically

$$\hat{\psi}(x) \to \varphi_2(x) . \tag{33}$$

That this is actually quite a natural choice for a Heisenberg type theory can be seen from the following consideration. In such a theory the equations of motion are given by

$$\delta\psi + l^2\gamma_5\lambda_\nu(\bar{\psi}\gamma_5\gamma_\nu\psi) = 0 . \tag{34}$$

With the choice (27), (33) for the asymptotic fields this equation of motion becomes asymptotically identical with the first Gürsey equation (26), provided that one puts $m = l^2/\lambda^3$. Also the second Gürsey equation is obtained asymptotically by applying the operator δ to equation (34). This gives

$$\Box\psi + \frac{l^2}{\lambda^3}\,\delta\hat{\psi} = 0\,, \tag{35}$$

the asymptotic form of which corresponds to the second equation (26). It is perhaps not idle to point out that the introduction of $\hat{\psi}$ does not represent a « linearization » of the Heisenberg equation. The essentially non linear nature of the theory is conserved in equation (31), which appears as a non linear constraint.

The present argument was intended to illustrate a mechanism which might conceivably lead to the derivation of massive particles from a theory of the Heisenberg type. The fact that in principle massive particles are not incompatible with an invariance principle of the type (1) has been clearly recognized by GÜRSEY. What can be claimed as new in the present discussion is the isolation of the possible forms of transformations compatible with (1) to which the asymptotical fields can be subjected, as well as the demonstration that there exist functions of ψ which transform in such a way that the contragredient asymptotic field φ_2 can be constructed.

5. – The significance of R.

In this Section it will be shown that the group defined by equation (1) is in no way directly connected to the non conservation of parity. One can indeed show that the set (φ_1, φ_2) of free Majorana fields can be used to form a complex spinor χ, which satisfies the free Dirac equation of a particle of mass m, that the quantum number N has exactly the same properties as the charge in the theory of the electron and that the reflection R can be broken up in two reflections, which can be identified respectively with C and P. This of course does not prove that invariance under C must generally exist in a theory of the Heisenberg type: C can only be defined by this method for the subspace of the total Hilbert space which corresponds to the particle which is described by the Majorana spinors φ_1 and φ_2.

The complex spinor χ can be defined by

$$\chi = \tfrac{1}{2}(1+\gamma_5)\varphi_1 + \tfrac{1}{2}(1-\gamma_5)\varphi_2\,; \qquad \chi^+ = \tfrac{1}{2}(1-\gamma_5)\varphi_1 + \tfrac{1}{2}(1+\gamma_5)\varphi_2\,. \tag{34}$$

It then follows from equations (28) and (29) that under (1) one must have

$$S_\alpha \chi S_\alpha^+ = \exp[+i\alpha]\chi \tag{35}$$

or its equivalent

$$[N\chi] = -\chi\,, \qquad [N\chi^+] = \chi^+\,. \tag{36}$$

The transformation (1) therefore becomes an ordinary gauge transformation applied to the complex spinor χ. The reflection properties of the fields φ_1 and φ_2 follow from (8) and the definition (31) of $\hat{\psi}$. One has

$$R\varphi_s(x)R^+ = i\gamma_4\varphi_s(\tilde{x})\,, \tag{37}$$

both fields φ_s therefore transform in the same way under R. Expressing (37) in terms of χ one obtains.

$$R\chi(x)R^+ = i\gamma_4\chi^+(\tilde{x})\,. \tag{38}$$

This is exactly the same as the reflection produced by CP in the theory of the electron, for which C and P can be defined by means of

$$C\chi(x)C^+ = \chi^+(x)\,, \qquad P\chi(x)P^+ = i\gamma_4\chi(\tilde{x})\,. \tag{39}$$

It must, however, be noted that the Heisenberg type theory described by equation (34) does not explicitly show a symmetry under the operation C. This would only be the case for a theory which is invariant under an exchange of ψ and $\hat{\psi}$. It follows that, though the theory is obvsiously invariant under CP, invariance under C may only be an approximate property appropriate to some sub space of the total Hilbert space.

* * *

Finally I would like to thank Dr. Lüders, Dr. Symanzik and Dr. Zumino for many interesting discussions on this subject.

RIASSUNTO

Si danno le definizioni di « una particella semplice » e di una « teoria del tipo Heisenberg ». Si mostra che in una tale teoria le particelle semplici hanno massa nulla. Per ottenere particelle di massa non nulla è necessario poter derivare dallo spinore della teoria due spinori asintotici che si trasformano in maniera complementare, cioè diversamente l'uno dall'altro sotto la trasformazione di Pauli.

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A Method of Reducing Radiation Damping in Nuclear Magnetic Resonance.

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Summary. — In high resolution nuclear magnetic resonance spectroscopy, radiation damping often distorts the line shapes. To reduce this effect, it has been found necessary to decrease parameters which are generally kept large from signal-to-noise considerations. It is suggested that by application of negative feedback to the resonant circuit containing the sample, the radiation damping effect may be reduced without deterioration in the signal-to-noise ratio.

1. – Introduction.

With the increasing resolution of nuclear magnetic resonance (n.m.r.) spectrometers, the influence of radiation damping, whose importance was first realized by Suryan [1], is being felt more and more strongly. The problem has been recently examined in detail by Bloembergen and Pound [2], Pfeifer [3], Bruce, Norberg and Pake [4] and Bloom [5].

The analysis shows that for a single coil spectrometer using a passive circuit, under slow passage conditions, the line width (measured between half-maximum

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(1) G. Suryan: *Curr. Sci.*, **18**, 203 (1949).
(2) N. Bloembergen and R. V. Pound: *Phys. Rev.*, **95**, 8 (1954).
(3) H. Pfeifer: *Ann. d. Phys.*, **15**, 311 (1955); *Suppl. Nuovo Cimento*, **6**, 1188 (1957).
(4) C. R. Bruce, R. E. Norberg and G. E. Pake: *Phys. Rev.*, **104**, 419 (1956).
(5) S. Bloom: *Journ. Appl. Phys.*, **28**, 800 (1957).

points) is given by

$$\Delta H = \frac{2}{|\gamma| T_2} + 4\pi f Q \chi_0 H_0 , \tag{1}$$

where f is the filling factor for the sample, Q is the quality factor of the coil containing the sample, χ_0 is the static nuclear susceptibility, H_0 is the value of the steady magnetic field, γ is the gyromagnetic ratio of the nuclei and T_2 is the inverse line width parameter. The second term on the right of equation (1) represents the contribution of radiation damping. If we assume the reasonable values $f = 0.05$ and $Q = 100$, then for protons in water at room temperature ($\chi_0 \simeq 3.2 \cdot 10^{-10}$/ml),

$$4\pi f Q \chi_0 H_0 \cong 2.0 \cdot 10^{-8} H_0 .$$

Thus radiation damping may often make a significant contribution to the line width in a spectrometer with a resolution of even 1 in 10^8, a resolution already achieved in many spectrometers.

The most expedient solutions to the problem of radiation damping have been felt ([6]) to be either to dilute the sample in a solvent or to reduce the sample volume, thus reducing χ_0 or f. Either procedure would obviously lead to a deterioration in the signal-to-noise ratio.

It is the purpose of this paper to show that a superior method of reducing radiation damping is to decrease effectively the quality factor Q of the coil containing the sample by the application of negative feedback.

2. – Reduction of Q by negative feedback.

Let negative feedback be applied to the tuned circuit through a resistor R_f from an amplifier of gain $-A$ and internal resistance R_i (see Fig. 1).

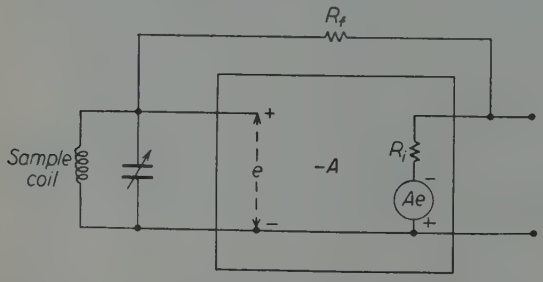

Fig. 1. – Provision of an effective shunt resistance to the coil by negative feedback.

It can be easily shown that the feedback has the effect of producing a resistance of value

$$\frac{R_f + R_i}{1 + A} ,$$

in shunt with the coil. The

([6]) R. B. Williams: *Ann. N.Y. Acad. Sci.*, **70**, 890 (1958).

effective shunt resistance of the tuned circuit at resonance reduces from R to R_e where

$$\frac{1}{R_e} = \frac{1}{R} + \frac{1+A}{R_f + R_i}, \tag{2}$$

leading to a reduction in the effective Q of the coil by the ratio R_e/R, *i.e.*,

$$Q_e = Q \cdot \frac{R_e}{R}. \tag{3}$$

3. – Noise considerations.

We will now calculate the noise output of the amplifier A without and with feedback (7).

CASE 1: *Noise output in the absence of feedback.* – The thermal noise generated in the shunt resistance R of the tuned circuit at resonance may be represented by a voltage generator V_n having a mean square voltage given by

$$[v_n^2]_{\text{mean}} = 4RkT\,\mathrm{d}f,$$

where k is the Boltzmann constant, T is the temperature of the resistor (°K) and $\mathrm{d}f$ is the bandwidth of the detecting system. The noise generated in the amplifier may be referred back to the input of the amplifier and treated as an equivalent voltage generator at the grid of the first tube having a mean square voltage given by

$$[v_{an}^2]_{\text{mean}} = 4R_{an}kT\,\mathrm{d}f,$$

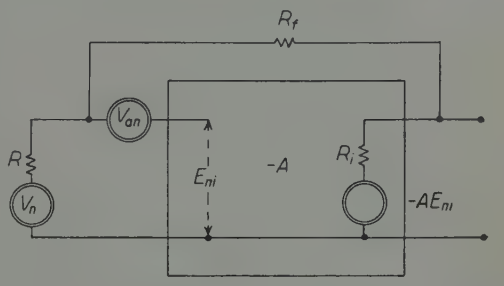

Fig. 2. – Equivalent circuit for noise in the case where Q is reduced by negative feedback.

Since there is no correlation between v_n and v_{an}, the r.m.s. noise voltage at the output of the amplifier is

$$N = A\sqrt{4(R+R_{an})\,kT\,\mathrm{d}f}. \tag{4}$$

CASE 2: *Noise output in the presence of feedback.* – The equivalent circuit for noise is given in Fig. 2. It can be shown that the r.m.s. noise voltage at

(7) C. N. W. LITTING: *Electronic and Radio Engineer*, **34**, 219 (1957).

the output of the amplifier is

$$N_{fb} = A\left[\left(\frac{R_e}{R}\right)^2 4RkT\,\mathrm{d}f + \left\{\frac{AR_e}{(1+A)R} + \frac{1}{1+A}\right\}^2 4R_{an}kT\,\mathrm{d}f\right]^{\frac{1}{2}}, \tag{5}$$

where R_e is defined by equation (2).

4. – Radiation damping and signal-to-noise ratio.

An attempt to decrease the radiation damping effect by reducing the filling factor f or the static susceptibility χ_0 (which we will call method 1) [6] has obviously no effect on the noise output of the amplifier which will be at the value N given by equation (4). From equation (1), we observe that a reduction in Q by negative feedback (the method suggested here, which we will call method 2) is equally effective in decreasing the radiation damping effect (it is the effective value Q_e, defined by equation (3), that should be used in equation (1)).

For the same decrease in the radiation damping effect, both method 1 and method 2 reduce the signal strengths in the same proportion. Hence the improvement in signal-to-noise voltage ratio achieved by using method 2 instead of method 1 is, from equations (4) and (5),

$$I \equiv \frac{N}{N_{fb}} = \left[\frac{1 + R_{an}/R}{(R_e/R)^2 + \{A/(1+A)\cdot(R_e/R) + 1/(1+A)\}^2 (R_{an}/R)}\right]^{\frac{1}{2}}.$$

Let us suppose that for the requirements of a particular experiment, the Q has to be reduced from 100 to 1 (*i.e.*, $R_e/R = 1/100$) and that $A = 100$, $R_{an} = 5R$. Then

$$I \cong 54\,,$$

which represents a considerable improvement. The method of reducing radiation damping described in this paper will be particularly useful when the r.f. field has to be maintained at a very low value to avoid saturation effects.

It may be added that the use of negative feedback to increase the bandwidth of the receiver in pulse apparatus for solids will have like advantages from the signal-to-noise viewpoint.

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The author wishes to express his thanks to Prof. R. S. KRISHNAN for his kind encouragement and to Dr. G. SURYAN for many helpful discussions. He is also grateful to the Department of Atomic Energy, Government of India, for the award of a research fellowship.

RIASSUNTO (*)

In spettroscopia per risonanza magnetica ad alta risoluzione, lo smorzamento della radiazione spesso distorce la forma delle righe. Per ridurre tale effetto è stato trovato necessario ridurre i parametri che, normalmente, si tengono elevati in considerazione del rapporto segnale-disturbo. Si suggerisce, per ridurre l'effetto di smorzamento della radiazione senza peggiorare il rapporto segnale-disturbo, di applicare una reazione negativa al circuito risonante contenente il campione.

(*) *Traduzione a cura della Redazione.*

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On a Full Geometrization of Conservation Laws in Gürsey's Formalism.

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Summary. — Gürsey's equations for the nucleon are investigated. Besides the generalized Pauli transformations C' a second transformation group C'' is introduced. From the unitary parts of C' and C'' an E_4-isospace is constructed. Isotransformations leave the conventional commutation relations invariant. It is shown that the baryon number and charge conservation laws are generated by rotations in E_4 and the isospin-vector $\boldsymbol{I}$ belongs to the (1, 2, 3) subspace of E_4.

Gürsey [1] has introduced Pauli transformations for equations of particles with non-zero mass. It is possible to obtain invariance of the equations with respect to these transformations, if we consider two 4-spinors fields ξ and χ together.

The system of equations:

$$\begin{cases} \gamma_\mu \partial_\mu \xi = -m\gamma_5 \chi \,, \\ \gamma_\mu \partial_\mu \chi = m\gamma_5 \xi \,, \end{cases} \tag{1}$$

where

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$

is invariant under

$$\begin{cases} \xi' = (a + c\gamma_5)\xi + (d + b\gamma_5)\xi^c \,, \\ \chi' = (a - c\gamma_5)\chi + (-d + b\gamma_5)\chi^c \,, \end{cases} \tag{2}$$

where a, b, c, d are arbitrary complex numbers.

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[1] F. Gürsey: *Nuovo Cimento*, **7**, 411 (1958).

We introduce a 2×4 wave matrix Ψ, which gives a common description of two Gürsey's 2×2 wave matrices:

$$\Psi = (\xi_1, -i\sigma_2\xi_2^*, \chi_1, -i\sigma_2\chi_2^*)\,, \tag{3}$$

where

$$\xi = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \qquad \chi = \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}.$$

Using Weyl's representation for the γ_μ-matrices:

$$\gamma_i = \varrho_2\Sigma_i\,, \qquad \gamma_4 = \varrho_1\,, \qquad \gamma_5 = \varrho_3\,,$$

we can write (1) in the following form:

$$D\sigma_2\Psi^*\Sigma_2\varrho_2 = m\Psi\,, \tag{4}$$

where $D = \partial_0 - \boldsymbol{\sigma}\cdot\boldsymbol{\Delta}$ and Σ_i, ϱ_i two sets of Dirac's matrices:

$$\Sigma_i = \sigma_i \times I\,, \qquad \varrho_i = I\times\sigma_i\,. \tag{5}$$

Transformations (2) are equivalent to

$$\Psi' = \Psi C'\,, \tag{6}$$

where

$$C' = \begin{pmatrix}S & 0\\ 0 & \sigma_2 S^*\sigma_2\end{pmatrix}, \tag{6'}$$

S, any 2×2 complex matrix, or

$$C' = i(\alpha_i + i\beta_i\varrho_3)\Sigma_i + \alpha_0 + i\beta_0\varrho_3\,. \tag{6''}$$

If we consider only unitary S

$$C' \equiv U' = i\alpha_i\Sigma_i + \alpha_0 \qquad \alpha_i^2 + \alpha_0^2 = 1, \tag{7}$$

we obtain the well-known Pauli transformations:

$$\begin{cases}\xi' = a\xi + b\gamma_5\xi^c\,,\\ \chi' = a\chi + b\gamma_5\chi^c\,,\end{cases} \qquad |a|^2 + |b|^2 = 1. \tag{8}$$

Besides U' Gürsey investigates a one-parameter phase factor group, which we denote U_3'':

$$\Psi'' = \Psi U_3'', \tag{8}$$

where

$$U_3'' = \begin{pmatrix} \exp[i\alpha]\cdot I & 0 \\ 0 & \exp[-i\alpha]\cdot I \end{pmatrix} = \exp[i\alpha\varrho_3]\,. \tag{9'}$$

We shall consider this transformation as a subgroup of a second unitary group U'':

$$U'' = i\lambda_i\varrho_i + \lambda_0 \qquad \lambda_i^2 + \lambda_0^2 = 1. \tag{10}$$

From (4) we see, that the field equation is invariant with respect to (10). U'' can be regarded as a unitary subgroup of a more general group C'':

$$C'' = i(\lambda_i + i\mu_i\Sigma_3)\varrho_i + \lambda_0 + i\mu_0\Sigma_3\,, \tag{11}$$

which we obtain from C' by interchanging Σ_i and ϱ_i.

Equation (4) is also invariant with respect to C''.

This new 8-parameter group is connected with transformations which mix in (1) ξ and χ. Using (3) one may easily see, that $\Psi'' = \Psi C''$ corresponds to

$$\begin{cases} \xi'' = (A + B\gamma_5)\xi + (C + D\gamma_5)\chi\,, \\ \chi'' = (-C + D\gamma_5)\xi + (A - B\gamma_5)\chi\,, \end{cases} \tag{12}$$

and $\Psi'' = \Psi U''$ is obtained from (12) by putting A, C real, B, D purely imaginary and $A^2 - B^2 + C^2 - D^2 = 1$.

Because U' contains only Σ_i and U'' only ϱ_i, these transformations commute. From two commuting unitary groups we can construct rotations in 4-dimensional Euclidean space $E_4 = U' \times U''$. This isospace includes the whole 4-parameter group, investigated by GÜRSEY (the 3-parameter Pauli transformations and the 1-parameter phase factor group).

If we introduce proton and neutron wave functions ψ_p, ψ_n by the following separation (2):

$$\begin{cases} \xi = \frac{1}{2}(1-\gamma_5)\psi_n - \dfrac{i}{2}(1+\gamma_5)\psi_p^c\,, \\ \chi = \frac{1}{2}(1+\gamma_5)\psi_n + \dfrac{i}{2}(1-\gamma_5)\psi_p^c\,, \end{cases} \tag{13}$$

(2) This choice is the most convenient for geometrical interpretation of charge and isospin. It differs from Gürsey's definition of proton and neutron by an interchange of ψ_p and ψ_n.

we shall have from (2) and (12)

$$C': \quad \begin{cases} \psi'_{\mathrm{p}} = \alpha\psi_{\mathrm{p}} + \beta\psi_{\mathrm{n}}\,, \\ \psi'_{\mathrm{n}} = \gamma\psi_{\mathrm{p}} + \delta\psi_{\mathrm{n}}\,, \end{cases} \tag{14}$$

$$C'': \quad \begin{cases} \psi''_{\mathrm{p}} = \lambda\psi_{\mathrm{p}} + \mu\psi^{c}_{\mathrm{n}}, \\ \psi''^{c}_{\mathrm{n}} = \nu\psi_{\mathrm{p}} + \tau\psi^{c}_{\mathrm{n}}\,, \end{cases} \tag{15}$$

U' and U'' are obtained from (14) and (15) by putting:

$$U': \quad \begin{cases} \alpha = \delta^* \\ \beta = -\gamma^* \end{cases} \quad \text{and} \quad U'': \quad \begin{cases} \lambda = \tau^* \\ \mu = -\nu^* \end{cases} \quad \text{respectively,} \tag{16}$$

where $|\alpha|^2 + |\beta|^2 = 1$ and $|\lambda|^2 + |\mu|^2 = 1$.

It is easily seen that the unitarity conditions (16) on C' and C'' are sufficient to preserve the conventional commutation relations.

If we take the general unimodular C' with $C'' = 1$ (or vice versa) we arrive at a 4-dimensional Minkowski's isospace. The invariance of commutation relations with respect to $U' \times U''$ distinguishes the E_4-group among other six-parameter isogroups.

An Euclidean isospace constructed from the product of unitary transformations (14) and (15) was investigated by D'ESPAGNAT [3]. In this space isovectors are linear combinations of nucleon wave function components and its conjugates.

It may be mentioned that in our $E_4 = U' \times U''$ isospace charge gauge transformation is a rotation O_{34} in the (3, 4) plane:

$$O_{34} = \exp i[(\Sigma_3 + \varrho_3)\alpha]\,. \tag{17}$$

Baryon number N corresponds to transformation (9). It is easily seen that U''_3 induces rotations in the (1, 2) and (3, 4) planes about the same angles, but with different signs.

From these considerations we obtain the Nishijma-Gell-Mann relation if the third isospin component I_3 corresponds to rotations O_{12} in the (1, 2) plane:

$$O_{12} = \exp\,[i(\Sigma_3 - \varrho_3)\beta]\,. \tag{18}$$

This transformation leaves ψ_{p} invariant and multiplies ψ_{n} by a phase factor. Thus it is seen that I_3 corresponds to neutron number conservation.

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[3] B. D'ESPAGNAT: *Nuovo Cimento*, **8**, 894 (1958).

The above geometrical considerations are a justification for treating the usual isospin-vector $\boldsymbol{I}$ as a isomomentum in the (1, 2, 3) subspace of E_4. $\boldsymbol{I}$ generates the following transformation group:

$$U = \exp[i(\boldsymbol{\Sigma} - \rho)\tau] \qquad \tau - \text{real.} \tag{19}$$

It is a three-parameter rotation group in usual E_3-isospace.

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The author is indebted to Prof. J. RZEWUSKI for helpful and valuable discussions.

RIASSUNTO (*)

Si esaminano le equazioni di Gürsey. Oltre alle trasformazioni di Pauli generalizzate C' si introduce un secondo gruppo di trasformazioni C''. Con le parti unitarie di C' e C'' si costruisce un isospazio E_4. Le isotrasformazioni non modificano l'invarianza delle relazioni di commutazione usuali. Si dimostra che le leggi di conservazione del numero di barioni e della carica sono dovute a rotazioni in E_4 e che il vettore di isospin $\boldsymbol{I}$ appartiene al sottospazio (1, 2, 3) di E_4.

(*) *Traduzione a cura della Redazione.*

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Differential Equations for the Renormalized Fields in the Point Source Lee-Model and Scalar Neutral Meson Theory.

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(ricevuto il 15 Aprile 1959)

Summary. — For two simple field theoretical models with infinite renormalization we show that the equations of motion can be formulated as differential equations for the renormalized fields involving only finite quantities. Thus the status of the initial value problem is not changed by the infinite renormalization. One has to observe that the field quantities at a sharp time, though not observables, have a well defined meaning as bilinear forms (infinite matrices). The equations involve certain limiting procedures which are closely related to Valatin's work on Quantum Electrodynamics but more explicit.

1. – Introduction.

It is generally believed that in Quantum Electrodynamics a space average of field quantities at a sharp time cannot be a finite observable. More concisely, let $\Phi(x)$ denote some component of the (renormalized) electromagnetic or matter field and $f(x)$ a sufficiently smooth c-number weight function. Then it is believed that

$$\Phi_f = \int \Phi(x) f(x) \, \mathrm{d}^4 x \,, \tag{1}$$

is a *bona fide* operator whereas

$$\Phi_f(t) = \int \Phi(\boldsymbol{x}, t) f(\boldsymbol{x}, t) \, \mathrm{d}^3 x \tag{2}$$

(*) On leave of absence from Istituto di Fisica Teorica, Università di Genova, and Istituto Nazionale di Fisica Nucleare, Sezione di Genova, with the support of a grant of the U.S. Government (P.L. 402 and P.L. 584).

is not. This is closely connected with the so-called infinite renormalization of the fields. The connection may be seen roughly as follows: The original, unrenormalized field $\Phi_u(x)$ was required to satisfy canonical commutation relations at equal times: *e.g.*, in the case of the matter field

$$\Phi_u^*(\boldsymbol{x}, t)\,\Phi_u(\boldsymbol{x}', t) + \Phi_u(\boldsymbol{x}', t)\,\Phi_u^*(\boldsymbol{x}, t) = \delta(\boldsymbol{x} - \boldsymbol{x}')\,. \tag{3}$$

Now the matrix elements of $\Phi_u(x)$ between physical states turn out to be proportional to $Z^{\frac{1}{2}}$, where Z^{-1} is a divergent quantity. Therefore one introduces the renormalized fields by

$$\Phi(x) = Z^{-\frac{1}{2}}\Phi_u(x) \tag{4}$$

and this quantity then has finite matrix elements between physical states. However, as a consequence of (3) and (4) we find that for any state at least one of the two expressions $\|\Phi_f(t)\Psi\|^2$ or $\|\Phi_f^*(t)\Psi\|^2$ must be infinite (proportional to Z^{-1}). This infinity may be traced to the fact that $\Phi_f(t)\Psi$ has too strong high energy components. One may get rid of these by averaging $\Phi_f(t)\Psi$ over some time interval as in (1) and this is the reason why (1) in contrast to (2) is a proper operator.

It would be of interest to know a finite equation of motion which is satisfied by the renormalized fields or even just to know of what general type such an equation could be. Does the necessity of a 4-dimensional smearing out of the fields imply that the equation of motion must be an integral rather than a differential equation with respect to time? If so, to what extent is the initial value problem affected? Is it for instance still true that the fields in an arbitrarily small time interval are a complete set of dynamical variables and determine the fields at any other time?

The answer to these questions can be worked out very explicitly for some simple models (Lee model [1]) and point source neutral scalar meson theory). These models have an infinite renormalization of field strength and mass although in other respects they are far from realistic field theories. We will find that it is possible to formulate these models in terms of differential equations for the renormalized fields [2]). No divergent quantities enter into the formulation but two somewhat unfamiliar features appear: The limiting processes (34), (68), and the auxiliary condition (37). The formulation of Quantum Electrodynamics by J. VALATIN [3]) suggests that the techniques which are

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[1]) T. D. LEE: *Phys. Rev.*, **95**, 1329 (1954)

[2]) Integral equations for the renormalized fields in the Lee model have been previously given by W. HEISENBERG [5]) and by M. FROISSART (private communication).

[3]) J. G. VALATIN: *Proc. Roy. Soc.*, A **226**, 254 (1954).

successful in the simple models treated here may be applied to more realistic theories and that our main result will carry over: The infinite renormalization does not affect the status of the initial value problem.

We are not concerned in this paper with the difficulties of physical interpretation in the point source Lee model which arise from the so-called « ghost states » ([4,5]), but only with a study of the mathematical formalism which is well defined. It seems likely, however, that the appearance of the auxiliary condition (37) is intimately connected with the « ghost state » problem.

2. – The renormalized Schrödinger equation in the lowest sector of the Lee model.

There is no need to describe the model here in detail. See ([1,4,5]). We restrict ourselves to the case of only one heavy particle which shall be at rest at the origin, and consider only θ-particles with angular momentum zero since the states of higher angular momenta are entirely uncoupled. We use the same notation as W. HEISENBERG. The Hamiltonian written in the unrenormalized quantities is

$$H = M_{\mathrm{V}}\psi_{\mathrm{V}}^{*}\psi_{\mathrm{V}} + \int \omega a^{*}(k)a(k)\,\mathrm{d}k - g_0\int \frac{k}{\sqrt{2\omega}}\left(\psi_{\mathrm{V}}^{*}\psi_{\mathrm{N}}a(k) + \psi_{\mathrm{N}}^{*}\psi_{\mathrm{V}}a^{*}(k)\right)\mathrm{d}k\,. \tag{5}$$

ψ_{V}^{*}, ψ_{V}, ψ_{N}^{*}, ψ_{N} are respectively the creation and destruction operators of a bare V- or N-particle at the origin. They satisfy the commutation relations ([6])

$$[\psi_{\mathrm{V}}, \psi_{\mathrm{V}}^{*}] = [\psi_{\mathrm{N}}, \psi_{\mathrm{N}}^{*}] = 1\,. \tag{6}$$

$a^{*}(k)$ and $a(k)$ are the creation and destruction operators of a θ-particle with angular momentum zero and energy $\omega = \sqrt{k^2 + m^2}$. The commutation relations are

$$[a(k), a^{*}(k')] = \delta(k - k')\,. \tag{7}$$

The formalism would be more elegant if we sticked consistently to the condition

$$\psi_{\mathrm{V}}^{*}\psi_{\mathrm{V}} + \psi_{\mathrm{N}}^{*}\psi_{\mathrm{N}} = 1 \tag{8}$$

([4]) G. KÄLLÉN and W. PAULI: *Dan. Mat. Fys. Medd.*, **30**, no. 7 (1955).

([5]) W. HEINSENBERG: *Nucl. Phys.*, **4**, 532 (1957).

([6]) Since we do not consider more than one heavy particle it is immaterial whether we treat these particles as bosons or as fermions.

and used the « isotopic spin » operators

$$\tau^{(+)} = \psi_{\mathrm{V}}^{*}\psi_{\mathrm{N}} ; \qquad \tau^{(-)} = \psi_{\mathrm{N}}^{*}\psi_{\mathrm{V}} ; \qquad \tau_3 = \psi_{\mathrm{V}}^{*}\psi_{\mathrm{V}} - \psi_{\mathrm{N}}^{*}\psi_{\mathrm{N}} , \tag{9}$$

rather than ψ_{V}, etc. We will not do this, however, but consider sometimes also states with no heavy particle, in particular the vacuum state $|0\rangle$, for which

$$\psi_{\mathrm{V}}|0\rangle = \psi_{\mathrm{N}}|0\rangle = a(k)|0\rangle = 0 . \tag{10}$$

The two constants M_{V} and g_0 are the parameters of the model. The expression (5) for the Hamiltonian makes sense only if the interaction is cut off at some high energy $\hat{E}$ *i.e.*, if some form factor $f(\omega)$ which is effectively 1 below $\hat{E}$ and 0 above $\hat{E}$ is inserted into the last term in (5). As we let $\hat{E}$ increase towards infinity we allow M_{V} and g_0 to change also, in order to keep the energies of the physical states as well as the effective interaction finite. A simple calculation ([1,4,5]), shows that the relevant quantity, which determines the energy spectrum and the scattering cross-section, is the function

$$h(z) = \frac{1}{g_0^2}(z - M_{\mathrm{V}}) + \int \frac{k^2\,\mathrm{d}k}{2\omega(\omega - z)} . \tag{11}$$

We can rewrite this as

$$h(z) = \alpha_1 + \alpha_2 z + z^2 \int \frac{k^2\,\mathrm{d}k}{2\omega^3(\omega - z)} , \tag{12}$$

with

$$\alpha_1 = \int \frac{k^2\,\mathrm{d}k}{2\omega^2} - \frac{M_{\mathrm{V}}}{g_0^2} , \tag{13a}$$

$$\alpha_2 = \frac{1}{g_0^2} + \int \frac{k^2\,\mathrm{d}k}{2\omega^3} . \tag{13b}$$

The last integral in (12) is convergent. Thus one must require that the two quantities α_1 and α_2 should be finite. They may be called the two renormalized parameters of the model. One sees from (13) that g_0 must approach zero on the imaginary axis as $\hat{E} \to \infty$ so that $-1/g_0^2$ diverges as $\log \hat{E}$, and that $-M_{\mathrm{V}}$ must approach infinity proportionally to $\hat{E}$. The fact that g_0 must be chosen imaginary is, of course, responsible for the appearance of the « ghost states ». But this does not concern us here.

The simplicity of the model lies in the fact that only transitions $\mathrm{V} \rightleftarrows \mathrm{N} + \theta$ are possible and therefore the particle numbers

$$n_1 = n_{\mathrm{V}} + n_{\mathrm{N}} ; \qquad n_2 = n_{\mathrm{V}} + n_{\theta} , \tag{14}$$

are constants of motion. In this section we consider the lowest non-trivial sector namely the case $n_1 = n_2 = 1$. An arbitrary state in this sector is described by a constant c (probability amplitude for a bare V-particle) and a wave function $\Phi(k)$ (probability amplitude for an N-particle plus a θ-particle of momentum k):

$$\Psi = c\psi_V^*|0\rangle + \int \Phi(k)\, a^*(k)\, \mathrm{d}k\, \psi_N^*|0\rangle \equiv \begin{cases} c \\ \Phi(k) \end{cases}. \tag{15}$$

We want to choose an imaginary g_0 to keep α_2 finite in the point source limit. Therefore the Hamiltonian is self adjoint with respect to the (indefinite) scalar product

$$(\Psi_1, \Psi_2) = \int \Phi_1^*(k)\, \Phi_2(k)\, \mathrm{d}k - c_1^* c_2 . \tag{16}$$

The eigenstates of H have simple orthogonality properties with respect to this scalar product ([5]) and it is therefore the definition (16) on which the physical probability interpretation must be based if such an interpretation can be given at all.

The Schrödinger equation

$$i\dot{\Psi} = H\Psi$$

becomes

$$i\dot{c} = M_V c - g_0 \int \frac{k\Phi(k)}{\sqrt{2\omega}}\, \mathrm{d}k , \tag{17a}$$

$$i\,\dot{\Phi}(k) = \omega\, \Phi(k) - \frac{g_0 c k}{\sqrt{2\omega}} . \tag{17b}$$

If we look at the properly normalized stationary solutions for finite energy we notice that in the limit of infinite cut-off they have the following two properties

$$C \equiv g_0 c \quad \text{is finite} . \tag{18}$$

$$\Phi(k) = \frac{kC}{\sqrt{2\omega}\,\omega} + \varphi(k) , \tag{19}$$

where $\varphi(k)$ decreases stronger than $\omega^{-\frac{1}{2}}$ for large ω. The same properties persist for all « physical states », *i.e.*, for all linear combinations of energy eigenstates with square integrable coefficient functions $f(E)$.

We rewrite the previous equations in terms of C and φ. The scalar product (16) becomes

$$(\Psi_1, \Psi_2) = \int \varphi_1^* \varphi_2\, \mathrm{d}k + C_1^* \int \frac{k\varphi_2}{\sqrt{2\omega}\,\omega}\, \mathrm{d}k + C_2 \int \frac{k\varphi_1^*}{\sqrt{2\omega}\,\omega}\, \mathrm{d}k + C_1^* C_2 \alpha_2 . \tag{20}$$

We see that for physical states not only C, and $\int |\varphi|^2 \mathrm{d}k$, but also $\int k\varphi/\sqrt{2\omega}\,\omega$ must be finite. In a formal fashion we can combine these three requirements by saying that the physical states correspond to the elements of an (ordinary, positive definite) Hilbert space in which the norm is defined as

$$\|\Psi\|^2 = \int |\varphi|^2 \,\mathrm{d}k + |C|^2 + \left| \int \frac{k\varphi}{\sqrt{2\omega}\,\omega} \,\mathrm{d}k \right|^2 . \tag{21}$$

This norm has, however, nothing to do with probabilities but serves only to define the concept of neighborhood of a state (topology). Probabilities are calculated by means of (20). Whenever necessary for the sake of clarity we can refer to (21) as the T-norm (T for topology) and to $\langle \Psi | \Psi \rangle$ as the P-norm (P for probability).

The first part of the Schrödinger equation becomes now

$$i\dot{C} = -g_0^2 \alpha_1 C - g_0^2 \int \frac{k\varphi(k)}{\sqrt{2\omega}} \,\mathrm{d}k . \tag{22}$$

Since H is an unbounded operator we must define yet its domain, *i.e.*, the subset of states on which it may be applied. Now, as $-g_0^2$ tends towards zero as $(\log \hat{E})^{-1}$, the integral in (22) may not be more than logarithmically divergent. This gives the *first domain condition* for H:

$$\lim_{k \to \infty} \sqrt{2}\,\omega^{\frac{3}{2}} \varphi(k) = B \qquad \text{shall be finite} . \tag{23}$$

Then (22) may be written according to (13*b*)

$$i\dot{C} = B - g_0^2 \left[\alpha_1 C + \alpha_2 B + \int (2\omega)^{-\frac{1}{2}} \left(\varphi(k) - \frac{Bk}{\sqrt{2\omega}\,\omega^2} \right) k \,\mathrm{d}k \right] . \tag{24}$$

Because of (23) the bracket on the right hand side is finite and, since g_0^2 approaches zero in the point source limit, the equation splits into two parts ([7]). We get from (24) and (17*b*) the final form of the Schrödinger equation

$$i\,\dot{C} = B , \tag{25a}$$

$$i\,\dot{\varphi}(k) = \omega \left(\varphi(k) - \frac{Bk}{\sqrt{2\omega}\,\omega^2} \right) , \tag{25b}$$

([7]) The argument that, in the limit $\hat{E} \to \infty$, (24) leads to the two equations (25*a*) and (26) rather than to (25*a*) alone is, of course, heuristic here. The real justification for (26) comes from the requirement that H should be self adjoint in the metric (20).

where B is defined by (23), and in addition the auxiliary condition

$$\alpha_1 C + \alpha_2 B + \int \left(\varphi(k) - \frac{Bk}{\sqrt{2\omega}\,\omega^2} \right) \frac{k}{\sqrt{2\omega}}\, \mathrm{d}k = 0 \,. \tag{26}$$

Equation (26) has to be regarded as a homogeneous boundary condition governing the selection of eigenvalues of H like the condition which one has to impose at the end points in the problem of a vibrating string of finite length. We will call it the *second domain condition* of H. In fact, just as in the case of the vibrating string, one can derive (26) from the requirement that the Hamiltonian, as defined by (25), shall be self adjoint with respect to the scalar product (20). Note that (26) is the only place in which α_1 enters into the renormalized equations. It is easy to see that the domain of H, *i.e.* the subset of states satifsying (23) and (26), forms a dense, linear subset in the Hilbert space (21).

Let us summarize! The original formulation of the model (for the sector $\mathrm{N}+\theta$) was given in terms of the Schrödinger equation (17) which had to be understood in the following way: Imagine the integral at the right hand side to be cut off at the upper limit $\hat{E}$, and the parameters M_{V} and g_0 to be such functions of $\hat{E}$ that in the limit $\hat{E} \to \infty$ the energy eigenvalues and the scattering cross-section stay finite. According to the preceding calculation we expect that the model can also be described in the following alternative way: The manifold of states is the Hilbert space (21). The Hamiltonian is defined by the Schrödinger equation (25) together with the domain conditions (23), (26). Probabilities are calculated by means of the scalar product (20). The two parameters of the model are the finite constants α_1 and α_2. We should still verify that this scheme is indeed adequate and gives us the same results for the energy eigenvalues and scattering cross-section as the original scheme which started from (5) or (17). This verification is simple but we will omit it here because it will be done in a more general way in the next section.

3. – Equations of motion for the renormalized fields in the Lee model.

There is no field strength renormalization for the N- or θ-fields in this model because there is no difference between « bare » and « physical » N- or θ-particles. But we have a renormalization for ψ_{V}. If Ψ is any physical state in the lowest sector one has

$$\langle 0 | \psi_{\mathrm{V}} | \Psi \rangle = c = \frac{1}{g_0} C \,. \tag{27}$$

To get finite matrix elements between physical states one therefore has to

introduce instead of ψ_{V} the renormalized quantity ψ_R by

$$\psi_R = g_0 \psi_{\mathrm{V}} \,. \tag{28}$$

We note that in accordance with the general remarks made in the introduction ψ_R^* is not a proper operator although it has finite matrix elements. For instance, one checks immediately that the application of ψ_R^* on the vacuum does not lead to a vector within the Hilbert space defined by (21). There are two alternative ways to give mathematical meaning to the symbol ψ_R^*. The first was already mentioned in the introduction and consists in an averaging over time. One then considers

$$\psi_R^*(t) = \exp\left[iHt\right] \psi_R^* \exp\left[-iHt\right], \tag{29}$$

as a distribution in the sense of Laurent Schwartz. In other words one asserts that for every smooth test function $f(t)$ there exists a proper operator $\psi_R^*(f)$ which formally corresponds to $\int \psi_R^*(t) f(t)\, \mathrm{d}t$. The other way is to regard ψ_R^* at a fixed time as a bilinear form in the Hilbert space of physical states (in plain language as an infinite matrix). The difference between such a bilinear form and an operator is that the former does not give a mapping from (a dense set of) the space into the space and therefore there is no eigenvalue problem associated with it. It does not correspond to a physical observable. This does not prevent us, however, from formulating the equations of motion in terms of bilinear forms. We can then retain the character of the field equations as differential equations in time also after renormalization, which has many obvious advantages. There is one difficulty which may be expected to appear in this context: The product of two bilinear forms has to be defined by matrix multiplication provided this gives a convergent result. In most cases of interest the straightforward product will be divergent and one has to use special limiting procedures to obtain a well defined quantity. In the case of the Lee model this difficulty is absent if one is interested only in the equations of motion. It appears only when one wants to express the Hamiltonian in terms of the renormalized fields at one time. We will not do this, but we shall discuss some such limiting procedures in the next section. In the present section all products which will appear in the final formulae are well defined as they stand.

After these preliminary remarks let us derive the renormalized field equations from the original ones in a heuristic fashion similar to the one used in the preceding section, and check afterwards that the resulting equations give us the same information about the physical quantities as the original scheme did. The unrenormalized equations, as derived

from (5), are

$$i\dot{\psi}_{\mathrm{V}} = M_{\mathrm{V}}\psi_{\mathrm{V}} - g_0\int \frac{k}{\sqrt{2\omega}}\psi_{\mathrm{N}}a(k)\,\mathrm{d}k\,, \tag{30}$$

$$i\dot{\psi}_{\mathrm{N}} = -g_0\int \frac{k}{\sqrt{2\omega}}a^*(k)\psi_{\mathrm{V}}\,\mathrm{d}k\,, \tag{31}$$

$$i\dot{a}(k) = \omega a(k) - g_0\frac{k}{\sqrt{2\omega}}\psi_{\mathrm{N}}^*\psi_{\mathrm{V}}\,. \tag{32}$$

Introducing ψ_R by (28) and replacing M_{V} according to (13a) the first equation becomes

$$i\dot{\psi}_R = -g_0^2\left[\alpha_1\psi_R + \int \frac{k}{\sqrt{2\omega}}q(k)\,\mathrm{d}k\right], \tag{30a}$$

with

$$q(k) = a(k)\psi_{\mathrm{N}} - \frac{k}{\sqrt{2\omega}\,\omega}\psi_R\,. \tag{33}$$

If we take a matrix element of (30a) between two physical states we see that the integral on the right may not be more than logarithmically divergent because the left hand side is finite and $1/g_0^2$ is logarithmically divergent. Therefore, the matrix elements of $\omega^{\frac{3}{2}}q(k)$ must have a high energy limit defining a bilinear form Q:

$$Q = \lim_{\omega\to\infty}\sqrt{2}\,\omega^{\frac{3}{2}}q(k)\,. \tag{34}$$

Using (13b) we then obtain

$$i\dot{\psi}_R = Q - g_0^2\left[\alpha_1\psi_R + \alpha_2 Q + \int \frac{k}{\sqrt{2\omega}}\left(a(k)\psi_{\mathrm{N}} - \frac{k}{\sqrt{2\omega}\,\omega}\psi_R - \frac{k}{\sqrt{2\omega}\,\omega^2}Q\right)\mathrm{d}k\right]. \tag{35}$$

The square bracket has finite matrix elements and, since $g_0 \to 0$ in the limit, (35) again splits into

$$i\dot{\psi}_R = Q \tag{36}$$

and the auxiliary condition

$$\alpha_1\psi_R + \alpha_2 Q + \int\left(\frac{k}{\sqrt{2\omega}}a(k)\psi_{\mathrm{N}} - \frac{k^2}{2\omega^2}\psi_R - \frac{k^2}{2\omega^3}Q\right)\mathrm{d}k = 0\,. \tag{37}$$

The other two field equations, *i.e.* (31), (32) need no modification other than

the replacement of $g_0\psi_V$ by ψ_R:

$$i\dot{\psi}_N = -\int \frac{k}{\sqrt{2\omega}} a^*(k)\psi_R \,\mathrm{d}k\,, \tag{38a}$$

$$i\,\dot{a}(k) = \omega a(k) - \frac{k}{\sqrt{2\omega}}\psi_N^*\psi_R\,. \tag{38b}$$

Equations (34) and (36) to (38) are the renormalized equations of motion. We want to show still that these equations constitute an adequate formulation of the model and lead to exactly the same consequences as the conventional approach starting from (5). Let us characterize an arbitrary state by the set of functions

$$\begin{cases} \Phi_n(k_1 \dots k_n) = \langle 0\,|\,a(k_1)\dots a(k_n)\psi_N|\Psi\rangle\,, \\ c_n(k_1 \dots k_n) = \langle 0\,|\,a(k_1)\dots a(k_n)\psi_R|\Psi\rangle\,. \end{cases} \tag{39}$$

From (36) and (38) we get the Schrödinger equation

$$i\dot{\Phi}_n(k_1 \dots k_n) = (\sum \omega_i)\,\Phi_n(k_i \dots k_n) - \sum \frac{k_i}{\sqrt{2\omega_i}}\, c_{n-1}(k_1 \dots k_{i-1}k_{i+1} \dots k_n)\,,$$

$$i\dot{c}_{n-1}(k_1 \dots k_{n-1}) = (\sum \omega_i)\, c_{n-1}(k_1 \dots k_{n-1}) + \langle 0\,|\,a(k_1)\dots a(k_{n-1})Q\,|\Psi\rangle\,.$$

For a stationary solution to energy E:

$$\Phi_n(k_1 \dots k_n) = R + \frac{1}{(\sum \omega_i) - E - i\varepsilon} \sum \frac{k_i}{\sqrt{2\omega_i}}\, c_{n-1}(k_1 \dots k_{i-1}\, k_{i+1} \dots k_n)\,, \tag{40}$$

$$\langle 0\,|\,a(k_1)\dots a(k_{n-1})Q\,|\Psi\rangle = (E - \sum \omega_i)\, c_{n-1}(k_1 \dots k_{n-1})\,, \tag{41}$$

where R is an arbitrary function containing a factor $\delta(E - \sum \omega_i)$. Now we multiply the auxiliary condition (37) from the left by $a(k_1)\dots a(k_{n-1})$, take the matrix element between Ψ and $|0\rangle$ and insert Φ_n and $\langle 0\,|\,a(k_1)\dots a(k_{n-1})Q\,|\Psi\rangle$ from (40), (41). This gives us the fundamental integral equation (122) of ref [5]:

$$h^{(+)}(E - \sum \omega_i)\, c_{n-1}(k_1 \dots k_{n-1}) + \int \frac{k}{\sqrt{2\omega}} R(k_1 \dots k_{n-1}, k)\,\mathrm{d}k + \tag{42}$$

$$+ \sum \frac{k_i}{\sqrt{2\omega_i}} \int \frac{k\,\mathrm{d}k}{\sqrt{2\omega}}\, \frac{c_{n-1}(k_1 \dots k_{i-1}k_{i+1} \dots k)}{(\omega + \sum \omega_k - E - i\varepsilon)} = 0\,.$$

From (34) we find still

$$\langle 0 \,|\, a(k_1) \dots a(k_{k-1}) Q \,|\, \Psi \rangle =$$

$$= \lim_{\omega \to \infty} \sqrt{2}\, \omega^{\frac{1}{2}} \left[\Phi_n(k_1 \dots k_{n-1}, k) - \frac{k}{\sqrt{2\omega}\, \omega} c_{n-1}(k_1 \dots k_{n-1}) \right], \tag{43}$$

and thus from consistency with the other equations

$$\lim_{\omega \to \infty} \omega^{\frac{1}{2}} c_{n-1}(k_1 \dots k_{n-2}, k) = 0 . \tag{44}$$

4. – Scalar neutral meson theory with fixed source.

In this model there is only one type of heavy particle. Apart from this the same notation can be used as in the previous sections. The formal Hamiltonian is

$$H = \int \omega a^*(k) a(k)\, \mathrm{d}k + M_0 \psi^* \psi + g \int \frac{k}{\sqrt{\omega}} \left(a(k) + a^*(k) \right) \mathrm{d}k \psi^* \psi . \tag{45}$$

We will treat the heavy particle as a fermion and require as commutation relations

$$\psi^2 = 0 ; \qquad \psi^* \psi + \psi \psi^* = 1 . \tag{46}$$

The restriction to nucleon number 0 or 1 is then automatic. The solutions of the Schrödinger equation are easily obtained. Introducing

$$N = \psi^* \psi , \tag{47}$$

$$b(k) = a(k) + g k \omega^{-\frac{3}{2}} N \tag{48}$$

we have

$$N^2 = N ; \qquad [N, b(k)] = 0 . \tag{49}$$

The commutation relations between the $b(k)$ and $b^*(k)$ are the same as those between the $a(k)$ and the $a^*(k)$ and the Hamiltonian becomes

$$\int b^*(k) b(k)\, \mathrm{d}k + MN , \tag{50}$$

with

$$M = M_0 - g^2 \int \frac{k^2}{\omega^2}\, \mathrm{d}k . \tag{51}$$

The model describes therefore only a free meson field and a fixed nucleon of mass M and there is no interaction between the nucleon and the mesons. While the physical content of the model is thus completely trivial, the discussion of the renormalization of the ψ-operator brings out a few points of interest. We first summarize briefly some remarks which are due to K. O. FRIEDRICHS ([8]). Let us imagine for the time being that we have a cut-off $\hat{E}$ appearing as an upper limit in all integrals and use the abbreviations

$$\xi = \int \frac{k^2}{\omega^3}\,\mathrm{d}k\,; \qquad B = \int \frac{k}{\omega^{\frac{3}{2}}}\,b(k)\,\mathrm{d}k\,. \tag{52}$$

The operator

$$U = \exp\left[-\tfrac{1}{2}\,g^2 N\xi\right]\exp\left[-\,gNB^*\right]\exp\left[gNB\right] \tag{53}$$

is unitary and has the property

$$U^{-1}b(k)\,U = a(k)\,; \qquad U^{-1}\chi U = \psi\,, \tag{54}$$

where χ is the destruction operator of the *physical* nucleon. U commutes with N, thus

$$\chi^*\chi = \psi^*\psi = N\,. \tag{55}$$

Using (53) we calculate the matrix element of ψ between the vacuum and the state of a physical nucleon and obtain

$$\langle 0\,|\psi|\,n\rangle = \exp\left[-\frac{g^2\xi}{2}\right]\langle 0\,|\,\chi\,\exp\left[-\,gB^*\right]\exp\left[gB\right]|\,n\rangle = \exp\left[-\frac{g^2\xi}{2}\right]. \tag{56}$$

Therefore the renormalized ψ has to be defined as

$$\psi_R = \exp\left[\tfrac{1}{2}\,g^2\xi\right]\psi\,. \tag{57}$$

One may ask whether

$$\psi_R(f) = \int \psi_R(t)\,f(t)\,\mathrm{d}t$$

is a proper operator in the limit $\hat{E}\to\infty$ if f is a smooth test function. Applying $\psi_R(f)$ on a state with one nucleon and a meson configuration Φ we get

$$\int f(t)\exp\left[-\,iMt\right]\exp\left[-\,g\int\frac{k\exp\left[i\omega t\right]}{\omega^{\frac{3}{2}}}\,b^*(k)\,\mathrm{d}k\right]\cdot \\ \cdot\exp\left[g\int\frac{k\exp\left[-\,i\omega t\right]}{\omega^{\frac{3}{2}}}\,b(k)\,\mathrm{d}k\right]|\,\Phi\rangle\,\mathrm{d}t\,. \tag{58}$$

([8]) K. O. FRIEDRICHS, private communication.

The norm squared of this expression is

$$(59)\qquad \int f^*(t_2) f(t_1) \exp\left[-iM(t_1-t_2)\right] \exp\left[g^2\varphi(t_1-t_2)\right] \cdot$$
$$\cdot \langle \Phi | \exp\left[g \frac{k}{\omega^{\frac{3}{2}}} \int b^*(k) \left(\exp[i\omega t_2] - \exp[i\omega t_1]\right) \mathrm{d}k \right] \cdot$$
$$\cdot \exp\left[-g \int \frac{k}{\omega^{\frac{3}{2}}} b(k) \left(\exp[-i\omega t_2] - \exp[-i\omega t_1]\right) \mathrm{d}k \right] | \Phi \rangle ,$$

where

$$(60)\qquad \varphi(t) = \int \frac{k^2}{\omega^3} \exp[i\omega t] \, \mathrm{d}k .$$

In the limit $\hat{E} \to \infty$, the function $\varphi(t)$ has a singularity at the point $t = 0$:

$$(61)\qquad \varphi(t) = -\log|t| + \text{regular part} .$$

It is this singularity which one hopes to render harmless in (59) by the time integration with the smooth weight function f. We see that the worst singularity in the integrand of (59) comes from the term in which both exponential operator functions are replaced by the identity since every b or b^* is associated with a factor which vanishes for $t_1 = t_2$. This singularity is according to (61) $|t_1 - t_2|^{-g^2}$. Therefore, we have the result that $\psi_R(f)$ is a proper operator if

$$(62)\qquad g^2 < 1 .$$

For $g^2 \geqslant 1$ a more judicious discussion is needed. One finds that even in that case $\varphi(t)$ is a distribution and hence $\psi_R(f)$ a proper operator ([8]).

Let us now discuss the field equations. For finite cut-off we obtain from (45)

$$(63)\qquad i\dot{a}(k) = \omega a(k) + g \frac{k}{\sqrt{\omega}} N ,$$

$$(64)\qquad i\dot{\psi} = M_0 \psi + g \int \frac{k}{\sqrt{\omega}} \left(a(k) + a^*(k)\right) \mathrm{d}k \, \psi .$$

In (64) we can first replace ψ everywhere by ψ_R. Secondly, we must look at the behaviour of the matrix elements of $a(k)\psi_R$ and $a^*(k)\psi_R$ between physical states for large values of k. If we take for the state on the left some meson configuration Φ_2, for the state on the right a nucleon plus a meson configuration Φ_1 we find

$$(65)\qquad \langle \Phi_2 | a(k)\psi_R | \chi^* \Phi_1 \rangle = \langle \Phi_2 | b(k) \exp[-gB^*] \exp[gB] | \Phi_1 \rangle =$$
$$= -\langle \Phi_2 | \psi_R | \chi^* \Phi_1 \rangle \frac{gk}{\omega^{\frac{3}{2}}} + \langle \Phi_2 | \exp[-gB^*] b(k) \exp[gB] | \Phi_1 \rangle .$$

The dependence of the second term on k is determined by the wave function of the mesons which compose Φ_1 and one can choose a dense set of states Φ_1 for which this term decreases with increasing k as fast as one likes. Therefore, the matrix elements of

$$a(k)\psi_R + g\frac{k}{\omega^{\frac{3}{2}}}\psi_R ,$$

will decrease fast with increasing k for a dense set of states. For $a^*(k)\psi_R$ we find

$$\langle \Phi_2 | a^*(k)\psi_R | \chi^* \Phi_1 \rangle = \langle \Phi_2 | b^*(k) \exp[-gB^*] \exp[gB] | \Phi_1 \rangle . \tag{66}$$

The right hand side is already Wick-ordered and we get no slowly decreasing contraction term as in (65).

Combining this information with (51) the equation (64) takes the final form

$$i\dot{\psi}_R = M\psi_R + g\int \frac{k\,\mathrm{d}k}{\sqrt{\omega}} (a(k) + gk\omega^{-\frac{3}{2}})\psi_R + \int \frac{k\,\mathrm{d}k}{\sqrt{\omega}} a^*(k)\psi_R . \tag{67}$$

The other equation of motion (*i.e.* (63)) needs no reordering, but we still have to define N in terms of ψ_R: This must be done by some limiting process. The most obvious one starts from (47) and (57) which are valid equations as long as the cut-off is finite. We get then

$$\langle\, |N|\, \rangle = \lim_{\hat{E}\to\infty} \exp\left[-g^2\int^{\hat{E}} \frac{k^2}{\omega^2}\,\mathrm{d}k\right] \cdot \tag{68}$$

$$\cdot \sum_n \frac{1}{n!}\int^{\hat{E}} \mathrm{d}k_1 \ldots \mathrm{d}k_n \langle\, |\psi_R^* a^*(k_1) \ldots a^*(k_1) \ldots a^*(k_0)\, |0\rangle \cdot \langle 0\, |a(k_1) \ldots a(k_n)\psi|\, \rangle .$$

A neater method is possible if one introduces the whole nucleon field $\psi(x)$ instead of the single operator ψ. The formal Hamiltonian is

$$H = \int \omega a^*(\boldsymbol{k}) a(\boldsymbol{k})\,\mathrm{d}^3k + M_0\int \psi^*(\boldsymbol{x})\psi(\boldsymbol{x})\,\mathrm{d}^3x + g'\int \frac{1}{\sqrt{\omega}} (a(\boldsymbol{k}) \exp[i\boldsymbol{k}\cdot\boldsymbol{x}] + \tag{69}$$

$$+ a^*(\boldsymbol{k}) \exp[-i\boldsymbol{k}\cdot\boldsymbol{x}])\,\psi^*(\boldsymbol{x})\psi(\boldsymbol{x})\,\mathrm{d}^3x\,\mathrm{d}^3k .$$

Introducing

$$\varrho(\boldsymbol{x}) = \psi^*(\boldsymbol{x})\psi(\boldsymbol{x}) ; \qquad \tilde{\varrho}(\boldsymbol{k}) = \int \varrho(\boldsymbol{x}) \exp[i\boldsymbol{k}\cdot\boldsymbol{x}]\,\mathrm{d}^3x \tag{70}$$

the previous formulae remain essentially unchanged. One only has to put

$\tilde{\varrho}(\boldsymbol{k})$ everywhere instead of N. The espression for $\varrho(\boldsymbol{x})$ in terms of ψ_R can be written as

$$\varrho(\boldsymbol{x}) = \lim_{y\to 0} \psi_R^*(\boldsymbol{x})\psi_R(\boldsymbol{x}+\boldsymbol{y}] \exp\left[-g^2\int \exp\left[i\boldsymbol{k}\cdot\boldsymbol{y}\right]\omega^{-3}\,\mathrm{d}^3k\right]. \tag{71}$$

This kind of limiting procedure was used by J. G. VALATIN in his formulation of the equations of Quantum Electrodynamics [3].

To summarize: The field equations can be written as differential equations involving only finite quantities. One has to observe that

a) The fields at one time are not operators but bilinear forms.

b) The terms appearing in the conventional equations as derived from the formal Hamiltonian have to be grouped together in a particular manner.

c) Some of the products between the field quantities have to be defined by a limiting procedure like (68) or (71).

d) In the case of the Lee model the field quantities at one time satisfy an auxiliary condition (if we have matrix elements between states which lie in the domain of H). We believe, however, that this latter phenomenon is peculiar to models with an indefinite metric.

RIASSUNTO

Viene mostrato come, in due semplici modelli di teorie di campo con rinormalizzazione infinita, le equazioni del moto possano essere formulate quali equazioni differenziali per i campi rinormalizzati: tali equazioni contengono solo quantità finite. Di conseguenza, la rinormalizzazione infinita non altera il carattere del problema dei valori iniziali (problema di Cauchy). Gli operatori di campo per un fissato valore del tempo, benchè non siano osservabili, hanno un significato matematico preciso come forme bilineari (matrici). Le equazioni contengono alcuni processi di passaggio al limite che sono strettamente connessi con quelli usati da Valatin in Elettrodinamica quantistica: noi diamo di essi una formulazione più esplicita.

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On Cauchy's Problem in General Relativity (*).

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Summary. — It is shown that a suitable set of Cauchy's conditions for Einstein's equations in vacuo consists in specifying the values of g_{kl} and $g_{kl,0}$ on a hypersurface $x^0=0$. The corresponding g_{0k} are to be found by solving three *spatial* second order differential equations, and g_{00} is then given by an *algebraic* relation. The first and higher time derivatives of $g_{0\mu}$ remain completely undetermined by the field equations, thus leaving room for arbitrary coordinate transformations. It is further shown that even if the g_{kl} are chosen close to their Galilean values, and if the $g_{kl,0}$ are small, the remaining $g_{\mu\nu}$ will in general not be close to their Galilean values. However, a detailed investigation of the physical components of the curvature tensor shows that the field can nevertheless be weak, thus implying that the large discrepancies between the $g_{0\mu}$ and their Galilean values are only a coordinate effect. The field is really strong only close to domains where the determinant of the $g_{\mu\nu}$ vanishes. By a suitable coordinate transformation, those domains can be made to shrink to points, and the resulting singularities may be interpreted as representing matter. This supports Einstein's view that matter should not be considered as something foreign to the metric field itself.

1. – Introduction.

It was recently shown by Papapetrou (1) that the metric can be asymptotically Minkowskian at spatial infinity only if it is asymptotically time-independent for $t \to \pm\infty$ (or at least can be brought into such a form by a suit-

(*) Partly supported by the U.S. Air Force through the Air Research and Development Command.

(1) A. Papapetrou: *Ann. d. Phys.*, **2**, 87 (1958).

able transformation). A similar result was found by PERES and ROSEN [2] who investigated the stability of a small time-dependent perturbation, and showed that if the perturbation lasts sufficiently long, it will grow to infinity at large distances from its sources. It was not clear, however, whether this instability of gravitational radiation fields has a real physical significance, or is only a co-ordinate effect, and also whether it still exists in a closed universe.

Moreover, Einstein's equations are hyperbolic ones (or at least can be brought into such a form for weak fields, by using harmonic co-ordinates), and therefore it is not quite natural to demand that their solution satisfy the Dirichlet or Neumann conditions at infinity. The mathematically correct formulation of the problem [3], which should give unambiguous results, necessitates boundary conditions of the Cauchy type: one gives the values of the field on a space-like hypersurface (by a suitable choice of the co-ordinates, we can take it as $x^0 = t = 0$) and the values of the first derivatives in a direction normal to this hypersurface (time derivatives). The solution of Cauchy's problem then consists in computing the values of the second time derivatives with the help of the field equations. Higher time derivatives can also be found by differentiating the field equations with respect to time. One can then write a series such as

$$g_{\mu\nu}(t) = g_{\mu\nu}(0) + g_{\mu\nu,0}(0) \cdot t + \tfrac{1}{2} g_{\mu\nu,00}(0) \cdot t^2 + \dots ,$$

which gives the desired solution in its domain of convergence.

We shall now attempt to find how the instability of gravitational fields arises from the solution of Cauchy's problem.

2. – Investigation of the Cauchy conditions.

For the sake of simplicity, we shall seek the solution of Cauchy's problem in a matter-free domain. It is convenient to write Einstein's equations in the form [4]

$$R^{00} - \tfrac{1}{2} g^{00} R = 0 , \tag{1}$$

$$R^0_k = 0 , \tag{2}$$

$$R_{kl} = 0 . \tag{3}$$

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[2] A. PERES and N. ROSEN: *Phys. Rev.* (to be published).

[3] P. MORSE and H. FESHBACH: *Methods of Theoretical Physics* (New York, 1953), p. 706.

[4] Greek indices run from 0 to 3, Latin indices from 1 to 3. A comma denotes partial differentiation. The Minkowski value of $g_{\mu\nu}$ will be denoted by $\eta_{\mu\nu}$.

It is well known (5) that this last equation directly gives $g_{kl,00}$ as a function of the $g_{\mu\nu}$, $g_{\mu\nu,0}$ and their spatial derivatives, while the four $g_{0\mu,00}$ are left undetermined by Einstein's equations. On the other hand, (1) and (2) imply that the $g_{\mu\nu}$ cannot be chosen arbitrarily.

Indeed one can easily show that

$$R^{00} - \tfrac{1}{2}g^{00}R \equiv -\tfrac{1}{2}g^{00}e^{kl}\,e^{mn}R_{kmnl}\,, \tag{4}$$

where

$$e^{kl} = g^{kl} - \frac{g^{0k}g^{0l}}{g^{00}}\,, \tag{5}$$

is the reciprocal matrix of g_{kl}, and therefore a function of the six g_{mn} only (6). It is highly convenient to work with this three-dimensional metric. Let us introduce three-dimensional covariant derivatives with the help of the affinity

$$\gamma^{k}_{mn} = \tfrac{1}{2}e^{kl}(g_{lm,n} + g_{ln,m} - g_{mn,l})\,. \tag{6}$$

These derivatives will be denoted by semi-colons. It is easily shown that

$$e^{kl}e^{mn}R_{kmnl} \equiv P + g^{00}(e^{kl}e^{mn} - e^{km}e^{ln})\Lambda_{kl}\Lambda_{mn}\,, \tag{7}$$

where

$$\Lambda_{kl} = \tfrac{1}{2}(g_{0k;l} + g_{0l;k} - g_{kl,0}) \equiv \tfrac{1}{2}(g_{0k,l} + g_{0l,k} - g_{kl,0}) - g_{0s}\gamma^{s}_{kl}\,, \tag{8}$$

and P is the scalar curvature of the metric g_{kl}.

Thus (1) is equivalent to

$$\frac{1}{g_{00} - e^{kl}g_{0k}g_{0l}} \equiv g^{00} = \frac{-P}{(e^{kl}e^{mn} - e^{km}e^{ln})\Lambda_{kl}\Lambda_{mn}}\,. \tag{9}$$

Hence g_{00} is given by an *algebraic formula*, once the other $g_{\mu\nu}$ are known. Moreover, one can show that

$$R^{0}_{m} \equiv g^{00}e^{kl}(R_{km0l} - e^{ns}g_{0s}R_{kmnl})\,, \tag{10}$$

$$\equiv \tfrac{1}{2}g^{00}e^{kl}\Big[(g_{0m,k} - g_{0k,m})_{;l} + g_{kl,0;m} - g_{ml,0;k} - \\ - 2e^{ns}g_{0s}P_{kmnl} - \Lambda_{kl}\frac{g^{00}{}_{,m}}{g^{00}} + \Lambda_{ml}\frac{g^{00}{}_{,k}}{g^{00}}\Big]\,, \tag{11}$$

(5) A. LICHNEROWICZ: *Théories Relativistes de la Gravitation et de l'Electromagnétisme* (Paris, 1955), p. 29.

(6) P. A. M. DIRAC: *Proc. Roy. Soc.*, A **246**, 333 (1958).

where P_{kmnl} is the curvature tensor of the metric g_{kl}. With the help of the relation

$$e^{ns} g_{0s} P^{\cdot}_{kmnl} \equiv g_{0l;km} - g_{0l;mk} , \tag{12}$$

one can bring (2) into the compact form

$$e^{kl}[(\sqrt{g^{00}}\Lambda_{km})_{;l} - (\sqrt{g^{00}}\Lambda_{kl})_{;m}] = 0 . \tag{13}$$

As $(g^{00}{}_{,m}/g^{00})$ is readily obtained from (9), we therefore get three second order partial differential equations for the three g_{0k}. These equations contain only spatial derivatives of g_{0k}, and they are linear in the second derivatives.

We see therefore that a convenient set of Cauchy's conditions consists in the values of the g_{kl} and $g_{kl,0}$ given on a hypersurface $x^0 = 0$. The corresponding g_{0k} are to be found by solving three *spatial* second order differential equations, and g_{00} is then furnished by an *algebraic* relation. The first (and higher) time derivatives of $g_{0\mu}$ remain completely undetermined by the field equations (7) and can be stipulated arbitrarily, thus leaving room for arbitrary co-ordinate transformations.

This agrees with the fact that in a Hamiltonian formalism (6), only the g_{kl} should be considered as canonical variables.

3. – Boundary conditions for g_{0k}.

We now have to specify suitable boundary conditions for g_{0k} in order to solve the three second order spatial differential equations which we have just derived. As none of the spatial co-ordinates is privileged, these equations must be elliptic, and the mathematically correct boundary conditions (3) are of the Dirichlet (or Neumann) type. If the domain of space which we are considering is closed—this includes the case of a part of a closed universe—we thus have to give the values of the g_{0k} (or of their normal derivatives) on its boundary. If the domain is open, we also have to give these values at infinity. The values of the g_{0k}—and hence of g_{00}—are thereby determined in the whole domain.

This is a very troublesome state of affairs, because we cannot stipulate any boundary conditions on g_{00}, and there is nothing to assure that the value of g_{00} will be « reasonable » (*i.e.* everywhere close to unity).

(7) It might be thoughtt hat $g_{0\mu,0}$ can be obtained by differentiating (9) and (13) with respect to time, and using (3) to eliminate $g_{kl,00}$ from the result. The final results however, are identities (the four identitites relating Einstein's equations), and nothing new can be got from this.

For instance, if we consider, as usual, the derivatives of $g_{\mu\nu}$ as small quantities of the first order, the denominator of the right-hand member of (9) is small of the second order. On the other hand, for an arbitrary choice of the g_{kl} (close to the first order of their Galilean values), P is in general of the first order. Thus g^{00} will be very large, and g_{00} close to zero.

We therefore see that in order to get a « reasonable field », the g_{kl} have to be chosen in such a way that P is small of the second order. (A similar restriction holds on $P_{,0}$, which restricts the freedom of choice of the $g_{kl,0}$.) However, P should not be too small: in particular, it should not be null, except in the case of a static universe.

Even when these conditions are realized, it is still impossible to state anything definite as to the value of g_{00}, if Dirichlet conditions are imposed on g_{0k}, because the value of g_{00} is rather determined by the *derivatives* of g_{0k}. If, on the other hand, we stipulate Neumann conditions, it will not be possible to assure that the g_{0k} themselves be small. There is still the possibility, if the domain we are considering is open, to subject the g_{0k} to Cauchy conditions. The value of g_{00} thus can be chosen close to unity, but only near the boundary. It is known [3] however, that the solution of an elliptic equation is then unstable, and there is no assurance as to the situation far from the boundary.

We thus conclude that *only exceptionally does the choice of quasi-Galilean* g_{kl} *lead to quasi-Galilean* $g_{0\mu}$.

4. – The linear approximation.

It is interesting to compare the situation discussed above with that arising when one assumes from the beginning that all the $g_{\mu\nu}$ and $g_{\mu\nu,0}$ are close to their Galilean values so that the linear approximation is valid. In the latter case, Eqs. (1), (2) and (3) go over respectively into:

$$(1a) \qquad \eta^{pq}\eta^{rs}(g_{pq,rs} - g_{pr,qs}) = 0 ,$$

$$(2a) \qquad \eta^{pq}(g_{pq,0m} - g_{0q,pm} - g_{mp,0q} + g_{0m,pq}) = 0 ,$$

$$(3a) \qquad \eta^{00}(g_{00,kl} - g_{0k,0l} - g_{0l,0k} + g_{kl,00}) + \eta^{pq}(g_{pq,kl} - g_{kp,lq} - g_{lq,kp} + g_{kl,pq}) = 0 .$$

The situation is now quite different. One cannot choose the g_{kl} and $g_{kl,0}$ arbitrarily on the surface $x^0 = 0$; they are subject to the restriction given by (1a) and by the equation

$$(4a) \qquad \eta^{pq}\eta^{rs}(g_{pq,0rs} - g_{pr,0qs}) = 0 ,$$

which is a consequence of either (1a) or (2a).

On the other hand, g_{00} is now entirely arbitrary, and g_{0k} is determined only to within the addition of a gradient. One must conclude that the linear approximation gives solutions which, as the limit of the solutions of the exact equations, represent a rather special case. One can expect therefore that conclusions based on the linear approximation are likely to be invalid in the general case.

5. – The semi-linear approximation.

We shall now assume, as usual, that the g_{kl} are close to their Galilean values, and that the $g_{kl,0}$ are small, so that terms quadratic in the derivatives of g_{kl} can be neglected. However, no such assumptions are made concerning $g_{0\mu}$. This is the reason why we call this approximation « semi-linear ».

Let us further suppose that $(\log g^{00})$ does not vary too rapidly, and let us consider once more Eq. (13). It is possible to make in a self-consistent way the same approximation on g_{0k} as we did on g_{kl}. Eq. (13) turns out to be, in this approximation, a *linear* inhomogeneous equation for g_{0k}, the inhomogeneous term being small of the first order. If the boundary values of g_{0k} are also small of the first order, g_{0k} will everywhere be small.

This need not be the case, however, if g^{00} is infinite, or close to zero.

6. – The physical components of the curvature tensor.

Keeping the assumptions of the previous section as to the smallness of $(g_{kl}-\eta_{kl})$, $g_{kl,0}$ and possibly of g_{0k}, we now intend to show that the large discrepancy that can arise between g_{00} and unity is merely a co-ordinate effect. In order to have an invariant measure of the strength of the field, we have to find the physical components of the Riemann-Christoffel curvature tensor [8], *i.e.* we have to compute

$$(14) \qquad R_{\varkappa\mu\nu\lambda} = \tfrac{1}{2}(g_{\varkappa\lambda,\mu\nu} + g_{\mu\nu,\varkappa\lambda} - g_{\varkappa\nu,\mu\lambda} - g_{\mu\lambda,\varkappa\nu}) + g^{\alpha\beta}(\Gamma_{\alpha\varkappa\lambda}\Gamma_{\beta\mu\nu} - \Gamma_{\alpha\varkappa\nu}\Gamma_{\beta\mu\lambda}),$$

(the $\Gamma_{\alpha\varkappa\lambda}$ belong to the complete $g_{\mu\nu}$ metric) and to contract its components with a tetrad of orthonormal vectors $h^{\mu}_{(\alpha)}$ such that

$$(15) \qquad g_{\mu\nu}h^{\mu}_{(\alpha)}h^{\nu}_{(\beta)} = \eta_{(\alpha\beta)}, \qquad \eta^{(\alpha\beta)}h^{\mu}_{(\alpha)}h^{\nu}_{(\beta)} = g^{\mu\nu},$$

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[8] F. A. E. Pirani: *Acta. Phys. Polon.*, **15**, 389 (1956).

where $\eta_{(\alpha\beta)}$ is Minkowski's matrix. That is, we have to compute

$$R_{(\alpha\beta\gamma\delta)} = R_{\varkappa\mu\nu\lambda} h^{\varkappa}_{(\alpha)} h^{\mu}_{(\beta)} h^{\nu}_{(\gamma)} h^{\lambda}_{(\delta)} , \tag{16}$$

and to examine whether these expressions are small or large. In general, the curvature tensor has 20 linearly independent components. However, in the present case, there are only ten, because of the ten Einstein equations

$$g^{\varkappa\lambda} R_{\varkappa\mu\nu\lambda} = 0 . \tag{17}$$

For instance, we may discard the six R_{0kl0}, because

$$R_{0kl0} = -(g^{00})^{-1}(R_{mkln} g^{mn} + R_{0kln} g^{0n} + R_{mkl0} g^{m0}) , \tag{18}$$

can be computed with the help of the other components, having only one null index, or none. If the latter are small, R_{0kl0} will also be small, unless g^{00} tends to zero.

Let us then examine $R_{\mu nst}$: If g^{00} is neither extremely large (which would make other $g^{\mu\nu}$ and hence $h^{\mu}_{(\alpha)}$ very large), nor close to zero, the only terms which can make $R_{\mu nst}$ large are those for which α or β or both are null, in equation (14). But none of these terms is quadratic in the first derivatives of g_{00}, so that they must contain terms small of the first order. Therefore $R_{\mu nst}$ is small.

If g^{00} tends to zero, it is not certain whether the g_{0k} are small or large, and we have to distinguish between two cases: If $|g_{0k}| \ll (g^{00})^{-\frac{1}{2}}$, we have $g_{00} \approx (g^{00})^{-1}$ and the largest terms which occur in $R_{\mu nst}$ arise from

$$g^{00}(\Gamma_{00t}\Gamma_{0ns} - \Gamma_{00s}\Gamma_{0nt}) .$$

Their order of magnitude is that of g_{0k}, which may be large. But this is to be multiplied by $h^{0}_{(\alpha)}$ the order of which is about $(g^{00})^{\frac{1}{2}}$, and the final result is small. (The same occurs for that part of $R_{(\alpha\beta\gamma\delta)}$ which is due to R_{0kl0}).

However, if $|g_{0k}|$ is not much smaller than $(g^{00})^{-\frac{1}{2}}$, the physical components of the curvature tensor may be large. They may also be large if g^{00} tends to infinity.

We thus conclude that the choice of quasi-Galilean values for g_{kl}, and small values for $g_{kl,0}$ is generally sufficient to obtain a locally weak field: the two exceptions occur when g^{00} tends to infinity, or when g^{00} tends to zero *and* $|g_{0k}|$ tends to infinity faster than $(g^{00})^{-\frac{1}{2}}$.

By a suitable choice of the co-ordinates we can get quasi-Galilean values for the other $g_{\mu\nu}$ throughout large domains. However it is nevertheless in

general impossible to realize this throughout the whole space with one single co-ordinate system. In other words, *even if the curvature of space-time is locally very small, its global topological structure is generally not flat.*

7. – An example: cylindrical waves.

Let us consider the metric [9]

$$ds^2 = e^{2\gamma-2\psi}(dt^2 - d\varrho^2) + e^{-2\psi}\varrho^2\, d\varphi^2 - e^{2\psi}\, dz^2 , \tag{19}$$

If

$$\psi_{\varrho\varrho} + \frac{\psi_\varrho}{\varrho} - \psi_{tt} = 0 , \tag{20}$$

and

$$\gamma_t = 2\varrho\psi_\varrho\psi_t , \qquad \gamma_\varrho = \varrho(\psi_\varrho^2 + \psi_t^2) , \tag{21}$$

this metric satisfies $R_{\mu\nu} = 0$. A convenient tetrad of orthonormal vectors is

$$h^t_{(t)} = e^{\psi-\gamma}, \qquad h^\varrho_{(\varrho)} = e^{\psi-\gamma}, \qquad h^\varphi_{(\varphi)} = \frac{e^\psi}{\varrho}, \qquad h^z_{(z)} = e^{-\psi}, \tag{22}$$

other components being null. One finds [10]

$$\begin{aligned}
R_{(z\varrho z\varrho)} &= -R_{(\varphi t\varphi t)} = e^{2\psi-2\gamma}(\psi_{\varrho\varrho} + 2\psi_\varrho^2 - \psi_\varrho\gamma_\varrho + \psi_t^2 - \psi_t\gamma_t) , \\
R_{(z\varphi z\varphi)} &= -R_{(\varrho t\varrho t)} = e^{2\psi-2\gamma}(\varrho^{-1}\psi_\varrho - \psi_\varrho^2 + \psi_t^2) , \\
R_{(ztzt)} &= -R_{(\varrho\varphi\varrho\varphi)} = e^{2\psi-2\gamma}(\psi_{tt} + 2\psi_t^2 - \psi_t\gamma_t + \psi_\varrho^2 - \psi_\varrho\gamma_\varrho) , \\
R_{(z\varrho zt)} &= -R_{(\varphi\varrho\varphi t)} = e^{2\psi-2\gamma}(\psi_{\varrho t} + 3\psi_\varrho\psi_t - \psi_\varrho\gamma_t - \psi_t\gamma_\varrho) .
\end{aligned}$$

One sees that if g_{00} tends to infinity, $R_{(\alpha\beta\gamma\delta)}$ tends to zero, but if g_{00} tends to zero, $R_{(\alpha\beta\gamma\delta)}$ is infinite. The fact that $R_{(\alpha\beta\gamma\delta)}$ tends to zero means that space-time is locally flat. It is not however, globally flat, but rather conical.

8. – Singular regions.

In general, we shall find regions for which g^{00} is positive, and other for which it is negative. The latter should be considered as unphysical. At the boundaries we have either g^{00} infinite (and the field is generally infinite) or

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[9] N. ROSEN: *Bull. Research Council, Israel*, **3**, 328 (1954).
[10] W. B. BONNOR: *Journ. Math. Mech.*, **6**, 203 (1957).

$g^{00} = 0$ (and the field may be finite or infinite). Moreover, the determinant of the $g_{\mu\nu}$ is there respectively null or infinite, so that by appropriate degenerate co-ordinate transformations, these boundaries can be made to shrink to points, thus eliminating all unphysical regions for which $g^{00} < 0$. In the vicinity of these points the field remains strong (if it was originally strong) because the $R_{(\alpha\beta\gamma\delta)}$ are scalars. These points should therefore be considered as real field singularities.

Although we have started from Einstein's equations in vacuo, there is a strong temptation to identify these singularities with point-masses. If this identification is correct, this last result supports Einstein's opinion that matter should not be considered as something foreign to the metric field itself ([11]).

([11]) A. EINSTEIN: *The Meaning of Relativity* (Princeton, 1953), p. 165.

RIASSUNTO (*)

Si dimostra che una serie adeguata di condizioni di Cauchy per le equazioni di Einstein per lo spazio vuoto consiste nello specificare i valori di g_{kl} e $g_{kl,0}$ su un'ipersuperficie $x^0 = 0$. I g_{0k} corrispondenti si debbono trovare risolvendo tre equazioni differenziali *spaziali* del second'ordine, e g_{00} è dato allora da una relazione algebrica. Le derivate rispetto al tempo, prima e superiori, di $g_{0\mu}$ restano completamente indeterminate dalle equazioni del campo, permettendo così arbitrarie trasformazioni di coordinate. Si dimostra inoltre che anche se le g_{kl} si scelgono prossime ai loro valori galileani e le $g_{kl,0}$ sono piccole, le restanti $g_{\mu\nu}$ non saranno, in generale, prossime ai loro valori galileani. Tuttavia, un esame dettagliato dei componenti fisici del tensore di curvatura mostra che il campo può non pertanto esser debole, il che fa pensare che le forti discrepanze tra le $g_{0\mu}$ e i loro valori galileani sono dovute solo alle coordinate adottate. Il campo è realmente forte solo in prossimità dei domini in cui si annulla il determinante delle $g_{\mu\nu}$. Con un'opportuna trasformazione delle coordinate, tali domini possono esser ridotti a punti, e le singolarità che ne risultano si possono interpretare come rappresentazioni di materia. Ciò corrobora l'opinione di Einstein che la materia non debba considerarsi come estranea al campo metrico stesso.

(*) *Traduzione a cura della Redazione.*

Gravitational Motion and Radiation - III (*).

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(ricevuto il 20 Aprile 1959)

Summary. — It is shown that the secular change of the total gravitational mass of a system of two bodies, uniformly rotating around each other, is exactly equal to minus the radiated energy (independently computed in the previous part of this paper).

In the previous part of this paper (1) we found the equations of motion of a system of two masses uniformly rotating around each other, and showed that the total (Newtonian) energy undergoes a secular change.

The same result can be obtained by computing the seventh order correction to the effective gravitational mass M. One has

$$-\sum \overset{*}{m}_7 = \sum \overset{*}{m}_5 \overline{\overset{*}{F}{}^0_{00}} + \sum m v^k v^l \overline{\overset{*}{F}{}^0_{kl}} + 2\sum m v^k \overline{\overset{*}{F}{}^0_{0k}} + \sum m \overline{\overset{*}{F}{}^0_{00}} ,$$

with

$$\begin{cases} \overset{*}{F}{}^0_{kl} = \frac{1}{4}\delta_{kl}(\overset{*}{\mathfrak{g}}{}^{00}_{5},_0 - \overset{*}{\mathfrak{g}}{}^{mm}_{5},_0) + \frac{1}{2}\overset{*}{\mathfrak{g}}{}^{kl}_{5},_0 + \frac{1}{2}(\overset{*}{\mathfrak{g}}{}^{0k}_{6},_l + \overset{*}{\mathfrak{g}}{}^{0l}_{6},_k) , \\ \overset{*}{F}{}^0_{0i} = -\frac{1}{2}(\overset{*}{\mathfrak{g}}{}^{00}_{7},_k + \overset{*}{\mathfrak{g}}{}^{00}_{5}\mathfrak{g}^{00}_{2},_k) , \\ \overset{*}{F}{}^0_{00} = \frac{1}{4}\overset{*}{\mathfrak{g}}{}^{ll}_{7},_0 - \frac{3}{4}\overset{*}{\mathfrak{g}}{}^{00}_{7},_0 + \frac{1}{4}\overset{*}{\mathfrak{g}}{}^{0k}_{6}\mathfrak{g}^{00}_{2},_k - \frac{3}{4}(\underset{(5)}{\mathfrak{g}^{00}\overset{*}{\mathfrak{g}}_{00}}\mathfrak{g}_{00})\mathfrak{g}^{00}_{2},_0 . \end{cases}$$

(*) Partly supported by the U.S. Air Force, through the Air Research and Development Command.

(1) A. Peres: *Nuovo Cimento* **11**, 644 (1959), hereafter refered to as II. All notations thoughout the present paper are those of II.

As the motion is circular, the kinetic and potential energies are separately constant (in the Newtonian approximation) and one has $\overset{*}{m}_5 = \overset{*}{g}^{00}_5 = \overset{*}{g}^{kk} = 0$. Moreover

$$\left(\frac{1}{4}\overset{*}{g}^{ll}_{7,0} - \frac{3}{4}\overset{*}{g}^{00}_{7,0}\right) \equiv \frac{d}{dt}\left(\frac{1}{4}\overset{*}{g}^{ll}_{7} - \frac{3}{4}\overset{*}{g}^{00}_{7}\right) - v^k\left(\frac{1}{4}\overset{*}{g}^{ll}_{7,k} - \frac{3}{4}\overset{*}{g}^{00}_{7,k}\right).$$

The first term in the right-hand member can be neglected, as its time average vanishes. There remains

$$(1) \quad -\sum \dot{\overset{*}{m}}_7 = \tfrac{1}{2}\sum m v^k v^l (\overset{*}{q}^{0k}_{6,l} + \overset{*}{q}^{0l}_{6,k} + \overset{*}{q}^{kl}_{5,0}) - \tfrac{1}{4}\sum m v^k (\overset{*}{q}^{00}_{7,k} + \overline{\overset{*}{g}^{ll}_{7,k}}) + \\ + \tfrac{1}{4}\sum m \overline{g^{00}_{2,k}\overset{*}{q}^{0k}_{6}}.$$

However

$$\tfrac{1}{4}\sum m \overline{g^{00}_{2,k}\overset{*}{g}^{0k}_{6}} = \sum m a^k \overline{\overset{*}{g}^{0k}_{6}},$$

$$= \frac{d}{dt}\left(\sum m v^k \overline{\overset{*}{g}^{0k}_{6}}\right) - \sum m v^k \frac{d}{dt}\overline{\overset{*}{g}^{0k}_{6}}.$$

Once more, the first term may be neglected, and there remains

$$-\sum m v^k \frac{d}{dt}\overline{\overset{*}{g}^{0k}_{6}} = -\sum m v^k \overline{\overset{*}{g}^{0k}_{6,0}} - \sum m v^k v^l \overline{\overset{*}{g}^{0k}_{6,l}}.$$

The last term of this equation cancels the first and second terms in (1), and there remains

$$-\sum \dot{\overset{*}{m}}_7 = -\sum m v^k [-\tfrac{1}{2} v^l \overset{*}{q}^{kl}_{5,0} + \tfrac{1}{4}(\overline{\overset{*}{q}^{00}_{7,k} + \overset{*}{g}^{ll}_{7,k}}) + \overline{\overset{*}{q}^{0k}_{6,0}}].$$

Adding to the right-hand member the expression (having a null time-average)

$$\frac{1}{2}\frac{d}{dt}\sum m v^k v^l \overset{*}{g}^{kl}_{5} = \sum m v^k \left(\frac{1}{2} v^l \overset{*}{g}^{kl}_{5,0} + a^l \overset{*}{g}^{kl}_{5}\right),$$

$$= \sum m v^k (\tfrac{1}{2} v^l \overset{*}{g}^{kl}_{5,0} + \tfrac{1}{4}\overset{*}{g}^{kl}_{5} \overline{g^{00}_{2,l}}),$$

one gets

$$-\sum \dot{\overset{*}{\eta}}_{7}^{k} = -\sum mv^k[-v^l \overset{*}{\underset{5}{g}}^{kl}{}_{,0} + \tfrac{1}{4}(\overline{\underset{7}{g}^{00}{}_{,k} + \underset{7}{g}^{ll}{}_{,k}}) - \tfrac{1}{4}\overset{*}{\underset{5}{g}}^{kl}\overline{\underset{2}{g}^{00}{}_{,l}} + \overline{\overset{*}{\underset{6}{g}}^{0k}{}_{,0}}] ,$$

$$= -\sum mv^k \overset{*}{\underset{8}{g}}^{k} .$$

This is exactly equal to the radiated energy that was computed in II.

RIASSUNTO (*)

Si dimostra che la variazione secolare della massa gravitazionale totale di un sistema di due corpi rotanti uniformemente uno attorno all'altro è esattamente uguale all'energia irradiata cambiata di segno (calcolata indipendentemente nella prima parte del presente lavoro).

(*) *Traduzione a cura della Redazione.*

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

Possible Test of Conservation of Parity in Production of K-Mesons and Hyperons.

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(ricevuto il 13 Marzo 1958) (*)

In [1,2] the hypothesis was suggested of a general requirement of invariance of strong, electromagnetic and weak interactions only under time reversal T and it was shown that gauge invariance in quantum electrodynamics and isotopic invariance in meson theory lead to invariance of renormalizable interaction Lagrangians under space reflection P. Invariance under T of a renormalizable and isotopically invariant Lagrangian of interaction of K-mesons with baryons does not lead to the conservation of parity. In connection with this the investigation of conservation of parity in production of K-mesons and hyperons presents some interest.

Let us consider [3] reaction $\pi+\mathrm{N}\to\mathrm{Y}+\mathrm{K}$ with observation of the subsequent decay $\mathrm{Y}\to\mathrm{N}+\pi$ (Y is Λ or Σ hyperon). If in production of hyperon and K-meson, parity is not conserved then the polarization vector of the hyperon may be represented as follows

$$\boldsymbol{P} = A\boldsymbol{n} + B\boldsymbol{p} + C\boldsymbol{q}\,, \tag{1}$$

where $\boldsymbol{n}$ is a unit vector normal to the production plane, and $\boldsymbol{p}$, $\boldsymbol{q}$ are unit vectors in the production plane $(\boldsymbol{pq})=0$. A, B, C, are the functions of the initial energy of the π-meson and of $\cos\gamma$, where γ is the angle between the direction $\boldsymbol{k}_0$ of the initial π-meson and the direction of the hyperon. The transition matrix for the decay $\mathrm{Y}\to\mathrm{N}+\pi$ in the centre of mass system is

$$M = N + F\boldsymbol{\sigma k}\,, \tag{2}$$

(*) The publication of this paper has been belated owing to an infortunate accident.

(1) V. G. Soloviev: *Journ. Exp. Theor. Phys.*, **33**, 537, 796 (1957).
(2) V. G. Soloviev: *Nucl. Phys.*, **6**, 618 (1958).
(3) T. D. Lee and C. N. Yang: *Phys. Rev.*, **104**, 254 (1956).

where N and F are complex numbers and k is a unit vector in the direction of the emitted π-meson.

The distribution of the π-mesons from the decay of hyperons in the centre of mass system is of the form

$$w \sim (NN^* + FF^*) + (NF^* + FN^*)A \cos\vartheta + \\ + (NF^* + FN^*)(B \cos\varphi - C \sin\varphi) \sin\vartheta , \tag{3}$$

where $\cos\vartheta = (\boldsymbol{nk})$, and φ is an angle between $\boldsymbol{p}$ and the projection of $\boldsymbol{k}$ on the production plane. It may be easily shown that the term, proportional to A is responsible for the up-down asymmetry in respect of the plane of production.

Left-right asymmetry in distribution of the emitted π-mesons in respect of the planes determined by $\boldsymbol{n}$ and $\boldsymbol{k}_0$ in each point of decay after integration under the angle ϑ may be obtained in

$$\frac{w_l}{w_r} = \frac{2(NN^* + FF^*) - (NF^* + FN^*)(B \sin\alpha + C \cos\alpha)}{2(NN^* + FF^*) + (NF^* + FN^*)(B \sin\alpha + C \cos\alpha)} , \tag{4}$$

where $\cos\alpha = (\boldsymbol{k}_0\boldsymbol{p})$ (α depends on the angle γ, we can put $\alpha = 0$).

From (3) it can be seen that the terms proportional to B and C lead to a forward-backward asymmetry with respect to the planes determined in each decay point by the vector $\boldsymbol{n}$ and by the vector orthogonal to $\boldsymbol{k}_0$ and lying in the production plane. This asymmetry may be written:

$$\frac{w_f}{w_b} = \frac{2(NN^* + FF^*) + (NF^* + FN^*)(B \cos\alpha - C \sin\alpha)}{2(NN^* + FF^*) - (NF^* + FN^*)(B \cos\alpha - C \sin\alpha)} . \tag{5}$$

Asymmetries (4) and (5) in the system of the hyperon at rest appear only when parity is not conserved in strong interaction at the production of K-mesons and hyperons. It is clear that the asymmetry must not disappear when integration over the angle γ is carried out.

It is desirable to analyze the experimental data obtained in [4,5] with the aim to discover the asymmetry mentioned above. Note that similar asymmetries will appear in the distribution of π-mesons from the decay of hyperons, produced in the reaction $K^- + p \to Y + \pi$, if parity is not conserved in the interaction of K-mesons with nucleons and hyperons.

* * *

The author is grateful to Prof. B. Pontecorvo for useful discussions.

([4]) F. S. Crawford jr., M. Cresti, M. L. Good, K. Gottstein, E. M. Lyman, F. T. Solmitz, M. L. Stevenson and H. K. Tycho: *Phys. Rev.*, **108**, 1102 (1957).

([5]) F. Eisler, R. Plano, A. Prodell, N. Samios, M. Schwartz, J. Steinberger, P. Bassi, V. Borelli, G. Puppi, H. Tanaka, P. Waloschek, V. Zoboli, M. Conversi, P. Franzini, I. Mannelli, R. Santangelo and V. Silvestrini: *Phys. Rev.*, **108**, 1353 (1957).

On the Direct Production of an Electron Pair by a High Energy α-Particle.

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(ricevuto il 6 Aprile 1959)

Ciok *et al.* [1] have reported on a jet $0+14\alpha$ with an energy of $H=3.3\cdot 10^2$ TeV/nucleon. This event has been found in the I-stack.

The track of the primary α-particle of this jet in the stack from the edge of the stack to the jet was 12.49 cm long. After 10.82 cm an interaction of the α-particle occurred in which two minimum ionizing particles are produced.

We have determined the energies of these two particles by relative scattering measurements between the α-particle track and the two minimum tracks (*). Assuming that the tracks were produced by electrons or by π-mesons we determined the total energies H'_{electron} and H'_{pion}, respectively, given in the Table I. In this table are also given the spatial angles ϑ between the α-particle and the minimum tracks and, for the case of π-mesons, the momenta of the particles.

To interpret this event the following possibilities have been considered:

1) Direct production of an electron pair by the high energy α-particle in the electric field of a nucleon of the emulsion. This process was studied for instance by Bhabha [2]. We have used the formulae given by Rossi [3] to obtain the mean free path λ of an α-particle with an energy of $H=1.32\cdot 10^3$ TeV for the direct production of an electron pair of any energy. The value obtained was 3.9 cm in emulsion. Furthermore, an estimate of the mean free path of an α-particle with the observed energy to produce directly an electron pair of total energy $\geqslant 670$ MeV gives a value of $\lambda(\geqslant 670\ \text{MeV})=9.9$ cm. This mean free path corresponds to a probability of $W=0.7$.

2) The production of a bremsstrahlung quantum by the primary α-particle (and subsequent conversion into an electron pair). Using the relation of Christy and Kusaka [3,4] we

(1) P. Ciok, M. Danysz, J. Gierula, A. Jurak, M. Mięsowicz and J. Pernegr: *Nuovo Cimento*, **10**, 1409 (1957).

(*) The «noise» value was found to be $D_{\text{noise}}=(0.13\pm 0.02)$ μm.

(2) H. J. Bhabha: *Proc. Roy. Soc.*, A **152**. 559 (1935).

(3) B. Rossi: *High Energy Particles* (New York, 1952).

(4) R. F. Christy and S. Kusaka: *Phys. Rev.*, **59**, 405, 414 (1941).

TABLE I.

	ϑ (rad)	H'_{electron} (MeV)	H'_{pion} (MeV)	p'_{pion} (MeV/c)
Track 1	$0.35 \cdot 10^{-2}$	530 ± 130	560	540
Track 2	$1.1 \cdot 10^{-2}$	140 ± 20	220	170

found that the probability for the production of a γ-quantum with energy $\geqslant 670$ MeV was $W < 10^{-4}$.

3) Production of two π-mesons in a nucleon-nucleon interaction (jet). Using the data from the table we have computed the transversal momentum of each of the produced π-mesons to $p_\perp \approx 2$ MeV/c. Taking into account the experimental spectra of the transversal momentum of the secondary particles of high energetic jets ([5,6]) we have estimated a value of $\approx 10^{-4}$ for the probability that in a high energy jet only particles with transversal momenta $\leqslant 100$ MeV/c are produced. Consequently, the probability that there are produced only particles with transversal momenta $\leqslant 2$ MeV/c will be $\ll 10^{-4}$.

4) Other possible explanations of the observed event are the following: an He nucleus has been produced in a fragmentation outside the stack, for instance as a hyperfragment

$$^4\text{He}_\Lambda \rightarrow {}^4\text{He} + \pi^0,$$

as a ^{4}He nucleus in an isomeric state (γ-decay) or as an α-particle together with a π^0-meson, whose decay γ-ray produces the electron pair in almost exact coincidence with the α-particle. Furthermore, the event can be a chance coincidence of the paths of an α-particle and a γ-ray. The probabilities of all these cases are $< 10^{-6}$.

From the above discussion we believe that it is possible to exclude all explanations other than the direct pair production by the high energy α-particle. A similar event was described by SEEMAN and GLASSER ([7]).

* * *

The authors express their gratitude to the laboratories of Czechoslovakia, Hungary and the Soviet Union for the examination and measurement of the α-track in their plates of the I-stack.

([5]) B. EDWARDS, J. LOSTY, D. H. PERKINS, K. PINKAU and J. REYNOLDS: *Phil. Mag.*, **3**, 237 (1958).

([6]) O. MINAKAWA, Y. NISHIMURA, M. TZUZUKI, H. YAMANOUCHI, H. AIZU, H. HASEGAWA, Y. ISHII, S. TOKUNAGA, Y. FUJIMOTO, S. HASEGAWA, J. NISHIMURA, K. NIU, K. NISHIKAWA, K. IMAEDA and M. KAZUNO: *Suppl. Nuovo Cimento*, **11**, 112 (1959).

([7]) N. SEEMAN and R. G. GLASSER: *Nuovo Cimento*, **4**, 703 (1956).

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On the Radiation of Mesons with a Constant Transverse Momentum P_T in Cosmic Ray Jets (*).

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(ricevuto l'11 Maggio 1959)

Several authors [1-3] observed that in high energy nuclear interactions ($E \geqslant 1000$ GeV) mesons are produced with a constant transverse momentum $P_T \simeq 0.4$ GeV, and that $p = P_T/\sin\varphi$, where P is the momentum of the meson emitted at direction φ with respect to the primary. The momentum spectrum of produced mesons is $\mathrm{d}P/p^2$.

The purpose of this note is to point out the resemblance of this observation to the Čerenkov Radiation.

Let us assume that the primary particle, when traversing nuclear matter, induces mesonic waves along its path, with an amplitude $\exp[i\,n\,k\,r]$. Where k is the wave number of the meson and n its refraction index describing its nuclear interaction. The condition for constructive interference of the mesonic waves is $n = \cos\varphi$. Now, according to the optical model n is a complex number connected to the forward scattering amplitude $f(0)$ by the relation

$$n^2 = 1 + \frac{4\pi\nu f(0)}{k^2},$$

(ν is the density of scattering centers). In order to obtain a radiation law with constant transverse momentum $p = P_T/\sin\varphi$, $f(0)$ must be a real negative number independent of k. This means that the complex processes of production and absorption of the highly interacting mesonic waves, that propagate along the primary path, may be described by a constant real scattering amplitude $f(0)$.

In this case, we shall write

$$-4\pi\nu f(0) = k_T^2,$$

and

$$n^2 = 1 - \frac{k_T^2}{k^2} = \cos^2\varphi \text{ or } p = \frac{P_T}{\sin\varphi},$$

$$(p = \hbar k);$$

(*) This work has been sponsored in part by the Geophysics Research Directorate of the Air Force Cambridge Research Center, Air Research and Development command, United States Air Force under contract AF61(052)-58 through the European Office ARDC.

(1) S. Hasegawa, J. Nishimura and Y. Nishimura: *Nuovo Cimento*, **6**, 979 (1957).

(2) B. Edwards, J. Losty, D. H. Perkins, K. Pinkau and J. Reynolds: *Phil. Mag.*, **3**, 237 (1958).

(3) P. Ciok, T. Coghen, J. Gierula, R. Holynski, A. Jurak, M. Mięsowicz, T. Saniewska and J. Pernegr: *Nuovo Cimento*, **41**, 741 (1958).

which is the condition for radiating mesons with a constant transverse momentum p_T.

Similar results are obtained by attributing to mesons of the mesonic cloud an effective mass $M_{eff} = p_T/c$.

If we carry even further the analogy with Čerenkov radiation, the energy of the emitted radiation is $\sin^2 \varphi \, dE$. Using the relation $\varphi = p_T/p$ for mesons emitted at small angles, one finds a momentum spectrum dp/p^2 in good agreement with the observations of EDWARDS *et al.* [2] and HASEGAWA *et al.* [1].

This model predicts that at very high energy meson multiplicity is independent of primary energy, and increases slowly with nucleus size, as $A^{\frac{1}{3}}$.

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A Remark on the Paradox of Einstein, Podolsky and Rosen.

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(ricevuto il 12 Maggio 1959)

The *uncontrollable* and *unpredictable* interaction between observed objects and observing apparatus (in Bohm's terminology [1]) has been the most mysterious and unsatisfactory part of the Copenhagen interpretation of quantum mechanics and has become the focus of criticism of many authors. Recently, one « physical » model for this process of interplay between micro-observables and detectors with macroscopic properties was proposed by H. S. Green [2]. He described the measuring apparatus as the ensemble of harmonic oscillators, such as one has to describe the molecular motion in a crystal, and assumed the suitable effective interactions of detectors with observables, then gave a rather satisfactory mathematical formulation of the measurement processes. As well-known, within the framework of the orthodox interpretation of quantum mechanics, no paradox can arise in the hypothetical experiment of Einstein, Podolsky and Rosen [3,4]. It is the aim of this short note to show how to understand the EPR « paradox » in Green's theory.

For the sake of simplicity, we take the EPR paradox in the form described in Bohm's text book [1,5]; because, to this special example of the EPR paradox, we can directly apply Green's model for detectors. That is, we consider a molecule of total spin zero consisting of two atoms, each of spin one half. The wave function of the system therefore

$$\Psi(1,2) = \frac{1}{\sqrt{2}}[\psi_+(1)\psi_-(2) - \psi_-(1)\psi_+(2)], \tag{1}$$

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(1) D. Bohm: *Quantum Theory* (New York, 1951), chap. XXII.

(2) H. S. Green: *Nuovo Cimento*, **9**, 880 (1958). M. Taketani had also suggested the same idea from a philosophical view point. See *Problems of Dialectics* (in Japanese, Rigakusha, Tokyo, 1945), p. 35.

(3) A. Einstein, B. Podolsky and N. Rosen: *Phys. Rev.*, **47**, 777 (1935); hereafter referred to as EPR.

(4) N. Bohr: *Phys. Rev.*, **48**, 696 (1935); W. H. Furry: *Phys. Rev.*, **49**, 393, 476 (1936); D. Bohm: reference (1).

(5) D. Bohm and Y. Aharonov: *Phys. Rev.*, **108**, 1070 (1957).

where $\psi_-(1)$ refers to the wave function of the atomic state in which the particle 1 has spin $\frac{1}{2}\hbar$, etc. The two atoms are then separated by a method that does not influence the total spin. After they are separated enough so that they cease to interact, any desired component of the spin of the first particle 1 is measured. Then, because the total spin is still zero, it should immediately be expected that the same component of the spin of the other particle 2 is opposite to that of particle 1. In the orthodox interpretation, however, the effects of the disturbance on the observed system by the indivisible quanta connecting it with the measuring apparatus play essential roles in bringing about the realization of one of various mutually incompatible potentialities of microscopic objects. Since particle 2 does not interact with particle 1 or with the measuring apparatus, one can not explain why particle 2 realizes its potentiality for a definite spin precisely in the same direction as that of particle 1. Moreover, one cannot explain the fluctuations of the other two components of the spin of particle 2 as the results of disturbances due to the measuring apparatus. This constitutes the essence of the paradox of Einstein, Podolsky and Rosen.

According to Green's formalism, the initial situation of the total system before one makes the observation of the spin component of particle 1 may be described by the statistical matrix

$$P_i = \Psi(1, 2)\varrho_D(1)\Psi^*(1, 2)\,, \tag{2}$$

where $\varrho_D(1)$ is the statistical matrix of the detector [6] for particle 1 (in what follows we use the same notations as in Green's work with the additional suffixes 1 and 2 corresponding to particles 1 and 2, respectively). The « effective » interaction of the detector with particle 1 [eq. (18) of reference [2]]

$$V(1) = \frac{1+\sigma_3(1)}{2} V_+(1) + \frac{1-\sigma_3(1)}{2} V_-(1)\,, \tag{3}$$

causes a transition from the initial combined system to the final state with the statistical matrix

$$P = \exp[-iV(1)t]\tfrac{1}{2}[\psi_+(1)\psi_-(2)\varrho_D(1)\psi^*_+(1)\psi^*_-(2) + \psi_-(1)\psi_+(2)\varrho_D(1)\psi^*_-(1)\psi^*_+(2) - \\ - \psi_+(1)\psi_-(2)\varrho_D(1)\psi^*_-(1)\psi^*_+(2) - \psi_-(1)\psi_+(2)\varrho_D(1)\psi^*_+(1)\psi^*_-(2)]\exp[+iV(1)t]\,. \tag{4}$$

Through averaging the apparatus coordinates (by setting $x^{(1)}_{\pm}=x^{(1)\prime}_{\mp}$ and $y^{(1)}_{\pm}=y^{(1)\prime}_{\mp}$ and integrating with respect to the $x^{(1)}_{\mp}$ and $y^{(1)}_{\mp}$), the last two terms in eq. (4) vanish, since they are multiplied by the factor $\prod_i \lambda_i$ [eq. (21) of reference [2]] which is effectively zero under the same assumption as Green's, and terms

$$\ll P \gg = \tfrac{1}{2}\psi_+(1)\psi_-(2)\psi^*_+(1)\psi^*_-(2) + \tfrac{1}{2}\psi_-(1)\psi_+(2)\psi^*_-(1)\psi^*_+(2)\,, \tag{5}$$

alone remain. This expression is completely in accord with Bohr's argument [3] against the criticism of EPR. in which he asserted that the observing apparatus and what is observed form a single indivisibe combined system.

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[6] See reference [2], Sect. **2.**

Thus, we can show that there is no inconsistency in the quantum mechanical conclusion that such correlations as in the spin components of particle 1 and particle 2 exist, but, from the foregoing point of view which belongs essentially to the Copenhagen interpretation, there is no way to raise the question of what is their origin. The author agrees with the attitude ([5,7]) that a further new explanation of quantum theory in terms of deeper sub-quantum mechanical levels must be developed. In the present status, nevertheless, at which any sub-quantum mechanical phenomena corresponding to the uncontrollable interaction in the measurement process are entirely not obvious, it does not yet become a problem of physics.

([7]) D. BOHM: *Causality and Chance in Modern Physics* (London, 1957), chap. IV and also *Observation and Interpretation* (London, 1954), p. 33.

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On the Geometrization of Electromagnetism.

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The unified field theories which have been so far put forward (cfr. Bargmann [1]). all deviate in some way or other from the usual four-dimensional Riemannian manifold for gravitation. In the present note we shall show that electromagnetic field equations alone (excluding gravition) can be derived approximately from a simple field law in the frame-work of a four-dimensional Riemannian geometry. A hint along the same line was first given by Thirring [2] and was improved by Soudan [3].

The interval in configuration space-time associated with a charged test-particle and its equation of motion will be given respectively by

$$ds^2 = g_{\mu\nu}(x, k)\, dx^\mu\, dx^\nu , \tag{1}$$

$$\frac{du^\alpha}{ds} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0 . \tag{2}$$

The metric tensor and the corresponding geometry now depend on k, some parameter associated with the test-particle, and in the linearized approximation

$$g_{\mu\nu}(x, k) = \underset{(0)}{g_{\mu\nu}}(x) + k \underset{(1)}{g_{\mu\nu}}(x) , \tag{3}$$

$\underset{(0)}{g_{\mu\nu}}$ being the Minkowskian values.

Now, the field equation can be written as (velocity of light is taken as unity)

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 2\pi k (\sigma u_\mu u_\nu - \tfrac{1}{2} g_{\mu\nu} \sigma) , \tag{4}$$

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(1) V. Bargmann: *Rev. Mod. Phys.*, **29**, 169 (1957).

(2) H. Thirring: *Phys. Zeits.*, **19**, 204 (1918).

(3) R. Soudan: *Arch. Sci.*, **3**, no. 1, 5 (1950); the present work was done independently of Soudan.

where σ is the proper charge density of the external sources, and the right hand side of (4) is so chosen that the correct vector potential come out at the linearized approximation.

In the coordinate system where

$$\frac{\partial \underset{(1)}{g^{\alpha}_{\mu}}}{\partial x^{\alpha}} = \frac{1}{2}\frac{\partial \underset{(1)}{g^{\alpha}_{\alpha}}}{\partial x^{\mu}}, \tag{5}$$

the field equation (4) approximately gives

$$\frac{\partial^2}{\partial x^{\alpha}\partial x^{\alpha}} \underset{(1)}{g_{\mu\nu}} = -4\pi[\sigma u_{\mu}u_{\nu} + \underset{(0)}{g_{\mu\nu}}\sigma]\,. \tag{6}$$

Now, if we put $\frac{1}{2}\underset{(1)}{g_{44}} = \varphi$, $i\underset{(1)}{g_{j4}} = A_j$, $k = e/m$, then the equations (5), (6), (2) in quasi-static approximation yield respectively the following electromagnetic equations

$$\operatorname{div} \boldsymbol{A} + \frac{\partial \varphi}{\partial t} = 0\,, \tag{5'}$$

$$\Box \boldsymbol{A} = 4\pi\sigma \boldsymbol{u}\,. \tag{6a}$$

$$\Box\varphi = 4\pi\sigma\,, \tag{6b}$$

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}s} = -\frac{e}{m}\left[\operatorname{grad}\varphi + \frac{\partial \boldsymbol{A}}{\partial t} + (\operatorname{curl}\boldsymbol{u}\times\boldsymbol{u})\right]. \tag{2'}$$

From the exact solution of (4), the interval for an electron whirling round a heavy nucleus of charge Ze is furnished by

$$\mathrm{d}s^2 = -\left(1 - \frac{2Ze^2}{mr}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\varphi^2 + \left(1 - \frac{2Ze^2}{mr}\right)\mathrm{d}t^2. \tag{7}$$

The slight change in Coulomb potential implied by (7) gives rise to a level shift of the Hydrogen atom, which unfortunately does not agree with the Lamb-Retherford value.

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γ-Ray Spectroscopy of Artificial Radionuclides Collected by Air Filtration at the Ground.

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(ricevuto il 30 Maggio 1959)

One of the subjects of the researches in the 1957-58 Geophysical Year, decided by the Working-Group on the nuclear radiation of the C.S.A.G.I. (Utrecht, 22-26 January 1957) is the study of the artificial radioactivity of the atmospheric air.

This is very interesting from the point of view of the problem of the atmospherice irculation. This interest is due to the presence, now frequent in the atmosphere, of the radioactive products originated from the nuclear and thermonuclear explosions.

Since these nuclides have a considerable persistence and their distribution extends to all the layers of the atmosphere, they are particularly useful as tracers for studies of the air streams and the exchanges among the various layers.

The usual method of collection, because of its efficiency and simplicity, consists in the filtration of a known great quantity of air through a filter paper having suitable characteristics.

The active deposit so obtained consists of two components: the natural (short-lived decay products of $^{222}_{86}$Em(Rn) and of $^{220}_{86}$Em(Tn)) and the artificial radionuclides.

The first component does not last long; the decay products of the Rn have half-periods of some ten minutes, while the decay products of the Tn have half-periods of some hours; so that after about four days the activity of the deposit is essentially due to the artificial radionuclides. This activity evolves appreciably during months and years.

At present eight collection stations, under the direction of C.N.I.-A.G.I., are working in Italy. Another group of stations is under the direction of the Italian Air Force.

The measurement of the total β-activity is systematically performed by the centres of the two groups.

Moreover a matter of the highest

importance is to single out the radionuclides, at least the most important ones, constituting the active deposit. A good possibility for this analysis is given by γ-ray spectroscopy, by means of scintillation detectors. In fact nearly all the radionuclides in question are γ-ray emitters ([1,2]).

A 100 channel pulse height analyser type RIDL thanks to a generous grant of C.N.R. is working, at the present time, at the Istituto di Fisica Sperimentale del Politecnico in Turin. This analyser, after some months of careful test in order to insure the necessary accuracy in the interpretation of the experimental results, has now attained a remarkable performance.

The samples tested by us have been collected in Naples at the Gabinetto di Meteorologia e Oceanografia of the Istituto Universitario Navale, on filter paper Millipore, by filtration of about 100 m^3 of air. The installation of the apparatus and the collections have been effected diligently by Dr. A. De Maio Assistent at the same Institute.

Among the various samples the by far most active has been collected on 9-th March last and we have followed and analysed its activity from the 11-the March to the 24-th April.

The scintillation detector consists of a NaI(Tl) crystal ($1\frac{1}{2}$ in. $\times 1\frac{1}{2}$ in.) followed by a 53 AVP photomultiplier. The pulses coming from the photomultiplier are sent to the RIDL analyser. The stability of the apparatus was periodically verified using γ-rays of $^{137}_{55}$Cs and $^{60}_{27}$Co ($E = 0.661$; 1.17; 1.33 MeV); no adjustment whatever was necessary. The photoelectric peaks of these γ-rays were always on the same channels.

We have obtained the following results up to now.

The diagram of the inverse of the activity against the time is a straight line with a good approximation. We can deduce that the radionuclides are prevalently originated by a single event only; the extrapolation of the straight line to meet the time axis localizes the date of the event up to about forty days before the collection.

Table I.

Prominent peak (MeV)	Nuclides
0.36	$^{140}_{56}$Ba (0.310); $^{140}_{57}$La (0.328); $^{131}_{53}$I (0.364)
0.49	$^{140}_{57}$La (0.487); $^{103}_{44}$Ru (0.498); $^{140}_{56}$Ba (0.54)
0.75	$^{95}_{40}$Zr (0.722; 0.754); $^{95}_{41}$Nb (0.768)
0.82	$^{140}_{57}$La (0.815)
1.62	$^{140}_{57}$La (1.596)

The effective γ-ray spectra have been obtained in three sets of measurements performed during the time included between 11-th and 21-th March, 8-th and 16-th April, 17-th and 24-th April (corresponding each one to ten hours of effective measurements (total minus accidental)).

The field of energies examined is extended from 0.25 to 1.80 MeV.

In Table I we indicate the values of the energies corresponding to the ob-

(1) A. Scafati: *Rend. Ist. Sup. Sanità*, **21**, 818 (1958); *Minerva Nucleare*, **3**, 1, 16 (1959).

(2) P. Amadesi, A. Cervellati, C. Melandri and O. Rimondi: *VIII Convegno della Ass. Geof. Italiana* (Roma, Febbraio 1959).

served prominent peaks and the nuclides which we presume responsible for the γ-rays.

By comparing the γ-spectra obtained during the three above periods of time, we can deduce particularly a considerable increase of the relative intensity of the 0.75 MeV peak of the $^{95}_{40}$Zr and $^{95}_{41}$Nb γ-rays.

In fact this intensity doubles in this period of time. This behaviour conforms with Schumann's table ([3]). In this table we can observe that the total activity of $^{95}_{40}$Zr and $^{95}_{41}$Nb increases rapidly at least in the first 100 days after the explosion.

Moreover it appears, from the γ-spectra examination of the various samples, that the presence of $^{137}_{55}$Cs (peak at 0.66 MeV) traces is not to be excluded. This would indicate the presence of old-dated radioactive material together with that of recent date.

Further researches are in progress and the results will be published afterwards.

* * *

We will express our gratitude to Prof. E. PERUCCA for its kind interest and useful discussions.

([3]) G. SCHUMANN: *Beitr. z. Phys. d. Atmosphäre*, **30**, 189 (1958).

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On the Possibility of Energy Depending Symmetry Properties.

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(ricevuto il 30 Maggio 1959)

A few months ago there has been some rumor of a strong parity violation in phenomena involving strong interactions of strange particles. These rumors have not yet been confirmed but have given rise to some speculations [1] about the possibility of understanding in a simple manner this supposed strong parity violation in strong interactions and the fact that no strong interaction parity violation in low energy nuclear phenomena has been observed inside a rather high degree of accuracy. Although this degree of accuracy is not at all so high as to give rise to really strong difficulties even in the case in which is assumed that the violation in a high energy region is real, it is sufficiently high to justify some investigations of the general question whether it is possible that some symmetry property might just depend on energy being, for instance, rigorous for states belonging to energies sufficiently low and for high energy states.

Of course if no restriction on the model of the physical system is considered it is always possible to construct models which have such a property; for instance, one may introduce an interaction which depends on total strangeness and is zero when total strangeness is zero and is violating the assumed symmetry property; but such an interaction would also be non local and would violate, for instance, the CPT theorem. Such kind of model is most certainly completely academical. It is therefore necessary to put some restrictions to the models which one likes to consider. It is the purpose of this paper to point out that, with some quite natural restrictions, it is possible to give a simple argument to rule out the possibility of some energy dependent symmetry properties. Of course it will be supposed that the interaction can be only local. Further more it will be supposed that if there are states in which the symmetry property is valid it is always possible to perturb this state with a very small but finite external interaction *represented by an operator which commutes with the operator of the supposed symmetry* (for instance if the symmetry is parity commuting with parity) without destroying the symmetry property. This hypothesis is justified by the fact that no physical system is really isolated and there are always

[1] See. *f.i.* B. d'Espagnat and J. Prentky: *Nuovo Cimento*, **12**, 164 (1955).

small external perturbations (for instance in a nucleus there are always small perturbations produced by external electric fields). Therefore if the previous hypothesis is not right the symmetry property could not be rigorous in any actual physical system. Now it might be proved that if the previous hypothesis is valid, for a very small perturbation, it is still valid when the perturbation is increased as much as one likes. It is then possible to transform states of the system corresponding to low values of the energy into states of the system having an energy as high as one likes. Therefore the symmetry property must be independent from the energy. The details of the proof (which is, by the way, rather simple) will be published in a later paper.

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On the Possible Existence of Hyperfragments with Mass Number $A=6$.

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(ricevuto l'8 Giugno 1959)

Some recently observed hyperfragment decay can be most simply interpreted by assuming the existence of Λ-hyperfragments with mass number $A=6$. These hyperfragments are characterized by a nuclear core which has no bound state: indeed ^{5}He and ^{5}Li are known only as compound nuclei in nuclear reactions.

The aim of this note is to see if the existence of such hypernuclei agrees with the present knowledge of the forces acting between hyperons and nucleons [1,2].

In a first and surely very crude approximation, we shall consider the $A=6$ hyperfragments as two body systems composed of a distorted $^5\mathrm{He}_\Lambda$ plus a nucleon in a p-state. Of course the model does not allow the introduction of all possible correlations amongst the particles. In particular to neglect the Λ-nucleon correlations may lead to underestimate the interaction energy of this pair of particles. A much better approximation would be to consider a three body model composed of an α-particle, a nucleon and a Λ-particle. A variational calculation based on this model is however fairly involved.

Let us first consider $^6\mathrm{He}_\Lambda$. In the two body approximation its wave function is of the form

$$\psi_{^6\mathrm{He}_\Lambda} \sim f^J \psi_{^5\mathrm{He}_\Lambda} r(\exp[-\alpha r^2] + x \exp[-\beta r^2] + y \exp[-\gamma r^2]), \tag{1}$$

where f^J is an eigenfunction of the total angular momentum, r is the relative $^5\mathrm{He}_\Lambda$-neutron coordinate and $\psi_{^5\mathrm{He}_\Lambda}$ is the wave function for the core hyperfragment which we take to be of the form

$$\psi_{^5\mathrm{He}_\Lambda} \sim (\exp[-a\varrho^2] + z \exp[-b\varrho^2]) \cdot \varphi_\alpha \,. \qquad (\boldsymbol{\rho} = \boldsymbol{r}_\Lambda - \boldsymbol{r}_\alpha). \tag{2}$$

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(1) D. B. Lichtenberg and M. Ross: *Phys. Rev.*, **110**, 737 (1958).
(2) R. H. Dalitz and D. W. Downs: *Phys. Rev.*, **111**, 967 (1958).

Here φ_α is the α-particle wave function, corresponding to a nucleon distribution of gaussian shape with r.m.s. radius $R=1.44$ fermi. A variational calculation for the binding energy of ${}^5\mathrm{He}_\Lambda$ leads to the value $B_\Lambda=2.76$ MeV (with the values $a=0.06$, $b=0.32$, $z=2.6$) in good agreement with the experimental datum ($B_\Lambda=2.9$ MeV).

The neutron radial wave function must necessarily be rather complicated because of the expected small binding energy: indeed with a trial function without sufficient flexibility the error in the binding energy may be larger than the binding energy itself.

The existence of a strong $p_{\frac{3}{2}}$ resonance in the neutron-α scattering at 0.95 MeV suggests that the neutron total angular momentum in the hyperfragment should be $\frac{3}{2}$. The possible values for J are therefore 1 and 2. We shall assume $J=1$ because this state contains the largest amount of the Λ-nucleon singlet state, the weights of singlet and triplet states being proportional to 2 and 1, respectively.

For the Λ-nucleon potential we use the one suggested by DALITZ ([2]):

$$V_{n\Lambda}(r)=\left(\frac{3+\sigma_\Lambda\cdot\sigma_n}{4}V_p+\frac{1-\sigma_\Lambda\cdot\sigma_n}{4}V_s\right)\exp\left[-0.459\left(\frac{r}{\eta}\right)^2\right],$$

$$V_p=22\ \mathrm{MeV};\qquad V_s=76\ \mathrm{MeV};\qquad \eta=\frac{\hbar}{2m_\pi c}.$$

For the neutron-α potential we take

$$V_{\alpha n}(r)=-(V_1+\lambda V_2)\exp\,[-r/k]^2,$$

with

$$\lambda=\begin{cases}0 & \text{for } l=0,\\ -l-1 & \text{for } j=l-\frac{1}{2},\\ l & \text{for } j=l+\frac{1}{2}.\end{cases}$$

We have performed two different variational calculations taking for V_1, V_2 and k the values determined by SACK *et al.* ([3]) and by VAN DER SPUY ([4]), namely:

(*a*) $V_1=47.32$ MeV; $V_2=5.85$ MeV; $k=2.3$ fermi (ref. ([3])),

(*b*) $V_1'=50.89$ MeV; $V_2'=4.07$ MeV; $k'=2.15$ fermi (ref. ([4])).

Sack's values have been obtained from the proton-α scattering p-wave phase shifts; those of VAN DER SPUY, on the other hand, are obtained from the neutron-α scattering data and fit both s- and p-wave phase shifts.

The variational calculation for the binding energy B_n of the neutron in ${}^6\mathrm{He}_\Lambda$ has given the following results: with the choice (*a*) for V_1, V_2 and k: $B_N=-0.34$ MeV for the following values of the variation parameters $\alpha=0.085$; $\beta=0.275$; $\gamma=0.0155$; $x=2.7$; $y=0.08$; $a=0.06$; $b=0.31$; $z=3$. With the choice (*b*) the neutron is not bound for 0.48 MeV.

([3]) S. SACK, L. C. BIEDENHARN and G. BREIT: *Phys. Rev.*, **93**, 321 (1954).
([4]) E. VAN DER SPUY: *Nucl. Phys.*, **1**, 381 (1956).

These results show that B_n is strongly dependent on the range of the α-neutron potential. To improve this point calculations have been repeated for different values of k. The results are shown in Table I.

TABLE I.

k (fermi)	$V_1 + \lambda V_2$ (MeV)	B_n (MeV)
2.15	54.96	+ 0.48
2.20	54.36	+ 0.2
2.25	53.77	— 0.02
2.30	53.17	— 0.34
2.35	52.57	— 0.69
2.40	51.98	— 1.05
2.45	51.38	— 1.38

One can get a rough estimate of the binding energy for $^6Li_\Lambda$ by adding to B_n the proton Coulomb energy which varies from 0.6 to 1 MeV according to the value one takes for k.

In conclusion one may say that the existence of $^6He_\Lambda$ (and perhaps also of $^6Li_\Lambda$) may be compatible with the present knowledge of the forces acting between nuclear particles and hyperons. Presumably calculations with a more refined model can improve our results. However one would need for them a considerably more detailed knowledge of the α-neutron potential.

* * *

We would like to thank Prof. L. A. RADICATI for his kind interest in this work.

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Derivation of Rate Equations: Corrections and Further Comments.

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(ricevuto l'8 Giugno 1959)

The purpose of this note is to retract in large part a critique [1] of a theory of irreversible behavior in large quantum mechanical systems presented by L. Van Hove [2] and of a closely parallel one for classical systems presented by Brout and Prigogine [3]. We shall confine comments to the Van Hove theory, with the understanding that analogous remarks apply to the classical theory. Errors were pointed out to the author by Van Hove. In short, we now agree that the method of treating singular energy integrals is free from arbitrariness, and that the transition operator in the $\lambda^2 t$ limit ($\lambda \to 0$, $t \to \infty$ such that $\lambda^2 t = \text{const}$) is indeed unitary. We would like to append below a few remarks on this subject. Van Hove's paper is somewhat densely written and some results are not at all obvious; these remarks may be helpful in avoiding further misunderstandings like the ones contained in our paper.

1) The « general asymptotic formula » Ing. (3.16) is nothing else than the formula

$$\int_0^T \mathrm{d}t \exp\left[i\varepsilon t\right] \cong \pi\delta(\varepsilon) + i\,\mathcal{P}\left(\frac{1}{\varepsilon}\right); \tag{1}$$

therefore Ing. (3.16) is good without restriction as to treatment of the energy pole. Our statement to the contrary was a simple error. It is essential here that this formula can be applied only when $T \gg \delta E^{-1}$, where integrands $f(\varepsilon)$ vary appreciably (say, by amounts equal to their average values) only over intervals of size comparable to δE. This formula is actually applied also to integrals in which the condition $T \gg \delta E^{-1}$ is violated, *e.g.*, in passing from VH (4.2) to VH (4.3). This pro-

(1) R. Ingraham: *Nuovo Cimento*, **9**, 99 (1958), referred to hereafter by the prefix Ing.
(2) L. Van Hove: *Physica*, **21**, 517 (1955), referred to hereafter by the prefix VH.
(3) R. Brout and I. Prigogine: *Physica*, **22**, 621 (1956).

cedure is justified by the remark that « $t - t_1 \gg \delta E^{-1}$ holds for the great majority of values t_1 »; presumably the integrals for which $t - t_1$ is small and thus (1) cannot be used have vanishingly small weight and can be neglected. Our only comment is that this is a delicate point and we would welcome seeing a fuller mathematical proof of the validity of this step.

2) Van Hove's « separation ansatz » VH (2.3) cannot be made to yield the equivalent form Ing. (3.30) for a similar reason. Van Hove's statement that it is to hold « for any operator diagonal in the $|E\alpha\rangle$ representation » must not be taken literally. In fact, the only constants of the unperturbed motion A for which this ansatz is used are of the form $\exp[i(E - H_0)\tau]$. Thus to prove the equivalence of these two equations, one would have to « divide out » the exponential in an equation of the form

$$\int \mathrm{d}\varepsilon \exp[i\varepsilon\tau] f(\varepsilon) = 0 , \tag{2}$$

that is, infer $f(\varepsilon) = 0$. Here the energy integrands $f(\varepsilon)$ all vary very little over intervals much smaller than δE. The A's for which the separation ansatz is used mainly have $\tau \gg \delta E^{-1}$, therefore (2) is trivially true without implying $f(\varepsilon) = 0$ (which would mean the validity of the equivalent form Ing. (3.30)). There remains the small class of A's encountered with small τ, for which (2) is actually not satisfied. One would like to see a proof similar in spirit to the one referred to in Sect. **1** that the use of VH (2.3) for these values of τ also makes no difference to the end results.

3) One convenient way to check the unitarity of the limit transition operator $U(t)$ is as follows. Since $U(t)$ must be unitary, then $\langle U(t)\varphi | U(t)\varphi' \rangle = \langle \varphi | \varphi' \rangle$. But by VH (6.7) this implies

$$\int \mathrm{d}\alpha'' P_t(E\alpha''; E\alpha) = 1 , \tag{3}$$

where $P_t(E\alpha''; E\alpha)$ is given as an expression involving ascending powers of λ^2 by VH (6.8). One can now directly verify that the various coefficients $\int \mathrm{d}\alpha'' P_t^{(m)}(E\alpha''; E\alpha)$ of λ^2 in the resulting power series vanish for the lowest orders $m = 1, 2, \ldots$ These are then equivalent to the sequence of equations Ing. (3.17).

4) The transition from the mathematical idealization of a continuous energy spectrum to the physical case of finite systems (E discrete) is treated in a short section VH Sect. **7**. In view of the mathematical subtleties of this transition, we believe that this treatment could be considerably amplified without running the risk of redundancy. One might think that for finite systems a λt limit ($\lambda \to 0$, $t \to \infty$ such that $\lambda t = \text{const}$) instead of the $\lambda^2 t$ limit of the transition operator would exist, due to the contribution proportional to T from the term $\varepsilon_i = 0$ in the typical time integrated energy sum

$$\int_0^T \mathrm{d}t \sum_i \exp[i\varepsilon_i t] f(\varepsilon_i) . \tag{4}$$

But if the sum were approximated by an energy integral as in the continuous case, and if the general asymptotic formula (1) were then used for large T, this expression would have no part $\propto T$. The condition that these two forms of (4) be compatible would place an *upper* bound on t, and one would get a condition like $t = O(\delta_l E^{-1}) = O(N)$ for the formal validity of the $\lambda^2 t$ « limit » in the finite case. Here $\delta_l E$ is the average spacing of the discrete energy levels E for a fixed value of α, the other quantum numbers, and N, for a crystal say, would be the number of atoms in it.

Howeves we are now convinced that no such arguments placing an upper bound on t apply. The $|E\alpha\rangle$ are here (metastable) states for the perturbed system with level widths of order $\lambda^2 \Gamma$ say, *not* (stable) states for the unperturbed system whose level widths are of course zero. One can presumably identify certain expressions occurring in the limit transition operator as these various level widths (cfr. the discussion VH, p. 527). By the assumption VH (7.1)

$$\delta_l E \ll \lambda^2 \Gamma . \tag{5}$$

The ε_i in (4) are spaced by $\delta_l E$ hence one cannot distinguish the level $\varepsilon_i = 0$ from its nonzero nearest neighbors, and the term $\propto T$ does not arise.

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Diffusion Coefficient of ^{37}A in Liquid N_2.

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F. P. Ricci

Comitato Nazionale Ricerche Nucleari - Roma

(ricevuto il 13 Giugno 1959)

The dependence of the diffusion coefficient D in binary mixtures of perfect liquids upon their molecular parameters has been by us [1] suggested in the form:

$$(1) \qquad D = 9 \cdot 10^3 \frac{\sigma_1^2}{\sigma_{i1}} \sqrt{\frac{\varepsilon_{i1}}{m_{i1}}} \cdot \exp\left(-\frac{2.6\,\varepsilon_1}{kT}\right) \mathrm{cm^2/s},$$

where we have indicated with subscripts i and 1 the molecular parameters of the tracer and those of the abundant liquid respectively.

Formula (1) (the validity of which we intend to check with different mixtures of perfect liquids) has been deduced by experimental data concerning ^{40}A-N_2 and HT-N_2 diffusions. In the first experiment the concentrations of A in N_2 samples were performed with a Nier type mass spectrometer and in the second one HT concentrations in N_2 were measured by internal gas G.M. counters. Therefore, the diffusion ^{40}A-N_2 was measured from a 3% mixture while the diffusion HT-N_2 from one at about 10^{-7}%.

We have now investigated the value of $D(^{37}A\text{-}N_2)$, in order *a*) to check that no effect of concentration upon diffusion was present in the earlier data (the concentration of ^{37}A in N_2 is about 10^{-7}%); *b*) to avoid the systematic errors eventually due to the different methods of analysis.

The diffusion apparatus and the experimental procedure have been already described [1,2]; the diffusions were performed at 73.4 °K and at 71.70 °K. The analysis of the mixtures has been done with a system based on G.M. counters with radioactive filling, which requires some precautions, as the addition of tagged Nitrogen samples to the usual

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(1) G. Cini-Castagnoli, G. Pizzella and F. P. Ricci: *Nuovo Cimento*, **10**, 300 (1958).

(2) G. Cini-Castagnoli, A. Giardini and F. P. Ricci: *Nuovo Cimento*: in press.

Trost filling mixture can disturb the working conditions of the counters. We have verified that the plateau is shortened and the efficiency slightly lowered if the tagged Nitrogen pressure gets over 3 mm Hg in the counter and if the activity is higher than 3000 c/m. Therefore, performing the analysis of diffusion samples we had to keep beyond these limits, where the proportionality between pressure of tagged Nitrogen and counts per minute has been tested. This is shown in Fig. 1. With a simple calibration it is possible to determine ^{37}A concentration in the samples.

The values are in good agreement with the values previously obtained, as it is shown in the graph of Fig. 2.

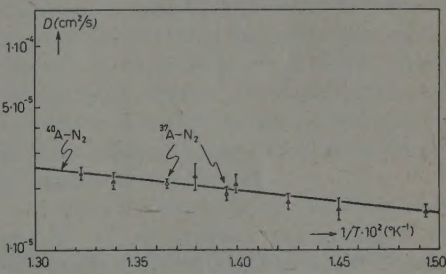

Fig. 2. – $D(^{40}A\text{-}N_2)$ and $D(^{37}A\text{-}N_2)$ vs. temperature.

TABLE I. – *Experimental results for* $^{37}A\text{-}N_2$ *diffusion.*

Run no.	Length of the capillary (cm)	Time (s)	Temperature (°K)	D $(cm^2/s)\cdot 10^5$
1	1.40 ± 0.02	3600	73.40 ± 0.10	2.15 ± 0.10
2	1.40 ± 0.02	3600	73.40 ± 0.10	2.05 ± 0.10
3	2.37 ± 0.02	9420	71.70 ± 0.10	1.90 ± 0.15

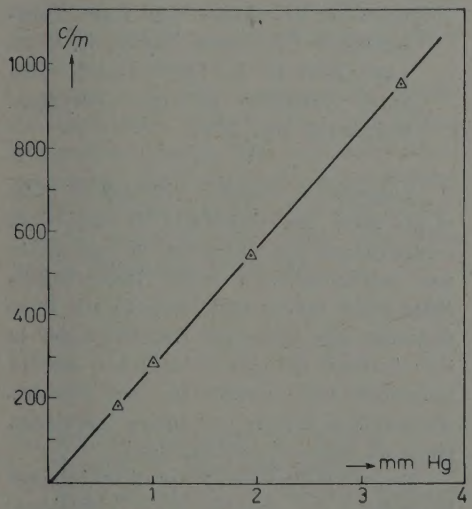

Fig. 1. – Counting rate vs. partial pressure of ^{37}A tagged sample.

Diffusion results are given in Table I.

We can therefore conclude that *a*) in our earlier experiments no effect of concentration was present; *b*) the two different methods of analysis for stable or radioactive isotopes give results for the diffusion coefficient which agree satisfactorily, thus not introducing any remarkable instrumental errors; *c*) the expression (1) as deduced by data concerning $^{40}A\text{-}N_2$ and $HT\text{-}N_2$ diffusions is confirmed.

Furthermore it is useful to point out that in simple liquids a tracer concentration of a few percent must be considered low at least in what concerns the diffusion process, in contrast with the behaviour observed in some solid mixtures [3].

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[3] D. LAZARUS: *Second United Nations International Conference* (Genèva, June 1958) A/Conf. 15/P/834.

LIBRI RICEVUTI E RECENSIONI

E. FERMI: *Termodinamica*, pp. VIII-180. Ed. Boringhieri. Lire 2500.

« Il presente libro è la raccolta delle lezioni da me tenute presso la Columbia University, New York, durante la sessione estiva del 1936. Esso costituisce un trattato elementare abbastanza completo, basato esclusivamente sulla termodinamica pura ... ». Così Fermi presenta questo volumetto.

La traccia del contenuto è classica: il concetto di sistema termodinamico, il primo e secondo principio, l'entropia, i potenziali termodinamici applicati alle reazioni dei gas ed alle soluzioni diluite, la costante dell'entropia. L'esposizione concisa ha conservato molto il sapore delle lezioni orali da cui il volume ha tratto origine, col naturale pregio della vivezza. Indubbiamente un corso cosiffatto risente della mancanza degli altri corsi, precedenti e collaterali, che furono sicuramente svolti insieme ad esso, e, quindi, benchè completo in sè, può essere studiato solo da chi già sappia di termologia.

La parte migliore del volume mi sembra essere l'introduzione ed enunciazione del primo principio che viene rigorosamente presentato come estrapolazione assiomatica del principio di conservazione dell'energia in meccanica.

Il volume, per la sua elementarità e per il tipo di applicazioni della termodinamica esposte, sembra diretto particolarmente a studenti universitari di chimica-fisica. Sarebbe stato desiderabile che un testo così limitato nel contenuto, dedicato a studenti e per essi utilissima lettura di complemento, fosse stato pubblicato in una edizione molto più economica; invece l'editore sembra aver preferito una veste tipografica ed una legatura più adatte alla monografia famosa che lo studioso ama avere in biblioteca.

All'editore va anche il biasimo per la presenza di alcuni errori di stampa.

G. C. MONETI

Expansion Machines for Low Temperature Processes. By S. C. COLLINS and R. L. CANNADAY. Oxford Library of the Physical Sciences, pp. 115.

L'ossigeno, l'azoto, l'idrogeno, l'elio e gli altri gas liquefatti a bassissima temperatura sono divenuti di così vasto uso nell'industria e nella ricerca scientifica, che molto utile appare un libro dedicato alle macchine frigorifere per la liquefazione dei gas. L'utilità è ancora maggiore poi quando in esso l'autore comunica al lettore una lunga esperienza ricca di brillanti realizzazioni.

Queste le idee con le quali ho iniziato la lettura del volumetto in questione. Purtroppo esso mi ha riservato una notevole delusione e ciò principalmente per due difetti essenziali.

In primo luogo il libro non affronta il problema nella sua interezza, ma è dedicato solo alle macchine ad espan-

sione, reciprocanti o a turbina, che costituiscono solamente una parte degli apparati di refrigerazione e del resto non sono l'unico mezzo per il raggiungimento delle basse temperature. Un compito così ristretto poteva essere risolto in uno o due brevi capitoli di un volume più ampio; per rendere compiuto il libro gli autori sono stati quindi costretti a completare l'esposizione con un cenno molto generico agli schemi delle macchine frigorifere. Ne risulta un complesso insoddisfacente, anche perchè alcuni problemi generali vengono impostati in modo decisamente strano alla luce dei moderni concetti fisici. Citiamo ad esempio le prime pagine sulla propagazione dell'energia.

In secondo luogo l'esposizione (eccezion fatta del capitolo sulle turbine) rifugge dalla sistematicità ed indulge invece molto alla rivista storica delle macchine costruite dal lontano 1828 ad oggi. Ciò ha portato varie pagine di dubbia utilità e al deterioramento del principale pregio del libro in quanto molte interessanti osservazioni e notizie sulla struttura e il funzionamento delle macchine ad espansione risultano sparpagliate in tutto il volumetto.

Proprio queste osservazioni, frutto dell'esperienza degli autori cui sopra accennavo, ritengo che costituiscano il principale pregio dell'opera e ne rendano la lettura utile a tutti coloro che progettano o semplicemente usano macchine per la liquefazione dei gas.

G. C. Moneti

A. Blanc-Caperre, P. Casal e A. Tortrat: *Méthodes mathématiques de la mécanique statistique*; Masson e Cie, Editeurs, Paris, pp. x-180; 3800 franchi.

Questa monografia sui fondamenti della meccanica statistica è il risultato della collaborazione tra un fisico, Blanc-Lapierre, e due matematici, Casal e Tortrat, della facoltà di Scienze di Algeri. Il volume è essenzialmente dedicato ai problemi di base della Meccanica Statistica. Quindi più che di metodi matematici si tratta di fondamenti matematici, e da questo chiarimento viene infatti alla mente la somiglianza nel contenuto e nelle tesi con la monografia di Khintchine: *Mathematical foundations of statistical mechanics*.

Diciamo subito che la via di attacco che gli autori seguono per giungere ad una formulazione coerente della meccanica statistica è quella solita, ormai definita dai risultati di Poincaré, Hadamard, Birkhoff e von Neumann. Si tratta prima di ammettere che le proprietà macroscopiche del sistema siano ricavabili dalle proprietà del sistema meccanico ad esso associato e che i risultati delle misure termodinamiche si esprimano mediante medie temporali. Si cerca allora di mostrare la possibilità di ricondurre queste medie temporali a medie nello spazio delle fasi. Alla base di questa possibilità sta, come è noto, l'ipotesi ergodica.

La monografia si compone di sei capitoli e di una bibliografia. Il primo capitolo è introduttivo e serve a presentare i principali problemi concettuali che verranno affrontati nel seguito. Nel secondo capitolo viene svolta una formulazione geometrica dello spazio delle fasi. Viene definita la misura della estensione invariante di fase, che dà l'elemento di integrazione ed è interpretabile quale probabilità elementare. Il capitolo terzo affronta la teoria ergodica, svolta secondo gli indirizzi di von Neumann e di Birkhoff. In particolare viene illustrata la condizione di indecomponibilità metrica per la varietà ad energia costante, necessaria affinchè le medie sul tempo coincidano con quelle in fase per una funzione sommabile qualsiasi. Questo studio conduce alla formulazione dell'ipotesi ergodica come ipotesi di transitività metrica, nozione che a sua volta si generalizza nella nozione di transitività topologica, nella

cui definizione e nelle cui conseguenze non entra più l'invarianza della misura in fase. Per chiarire la connessione con l'ipotesi ergodica ricorderemo che mentre la transitività topologica è equivalente, come si può mostrare, all'esistenza di almeno una traiettoria densa, la transitività metrica comporta che quasi tutte le traiettorie siano dense. I capitoli quarto e quinto trattano rispettivamente la meccanica statistica classica e la meccanica statistica quantica. L'appendice si riferisce essenzialmente al capitolo sulla teoria ergodica e contiene una esposizione succinta ma sufficiente della teoria della misura e alcune nozioni di topologia.

Ci sembra in conclusione che gli autori abbiano svolto un ottimo lavoro di esame critico dei fondamenti della meccanica statistica classica. È invece superficiale ed affrettata la trattazione della meccanica statistica quantica. In particolare non si trova cenno dei contributi dati in questo campo da von Neumann e da Pauli e Fierz, e di tutta la problematica sorta per giungere a una formulazione soddisfacente dell'ipotesi ergodica quantistica. Nè d'altra parte viene offerta alcuna trattazione del formalismo della meccanica statistica quantica. Riteniamo che il volume possa essere utile allo studioso di questioni ergodiche classiche. Può pertanto servire ad approfondire alcune delle nozioni insegnate nei nostri corsi di meccanica statistica.

R. GATTO

Direttore responsabile: G. POLVANI Tipografia Compositori - Bologna

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